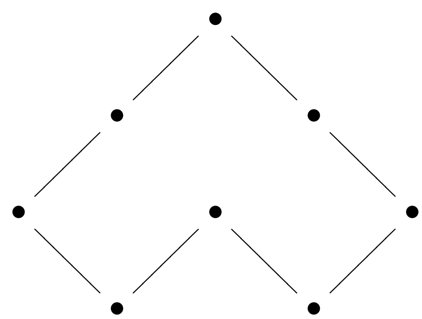
Question: Is it possible for two different ideals I, J to give isomorphic cyclic modules

$$R/I \cong R/J?$$

If so, we need to describe all the ideals J which give the same quotient up to isomorphism.

Cyclic modules can be visualized as a diagram with one peak:



For example, this could be $K[X, Y]/(X^3, X^2Y^2, Y^3)$. A finitely generated module could be visualized as a union of overlapping pictures of this kind.

[If someone can create the picture, I will insert it here.]

7. Products and coproducts

Today we talked about products and coproducts (sums) and some of the consequences of these concepts. We also talked about the difference. In particular, infinite products can be finitely generated but nontrivial infinite sums cannot.

(1) products
(2) sums
(3) projective modules
(4) finite generation

7.1. **products.**

If $M_\alpha, \alpha \in I$, is a family of R-modules then we can form the Cartesian product

$$\prod_{\alpha \in I} M_\alpha = \{(x_\alpha) \mid x_\alpha \in M_\alpha\}$$

This is an R-module with addition and action of R given coordinate-wise. We have projection maps onto each coordinate:

$$p_\alpha : \prod_{\alpha \in I} M_\alpha \to M_\alpha$$

These are R-module homomorphisms.

Theorem 7.1. *The Cartesian product $\prod_{\alpha \in I} M_\alpha$ is the product in the category of R-modules.*

Proof. The statement is that, given any module L and homomorphisms $f_\alpha : L \to M_\alpha$, there exists a unique homomorphism $f = \prod f_\alpha : L \to \prod M_\alpha$ making the following diagram commute:

$$\begin{array}{ccc} \prod M_\alpha & \xrightarrow{p_\alpha} & M_\alpha \\ \uparrow{\scriptstyle \exists! f} & \nearrow{\scriptstyle f_\alpha} & \\ L & & \end{array}$$

This is obviously true: f must send $x \in L$ to the unique element of $\prod M_\alpha$ whose α coordinate is $f_\alpha(x)$. □

7.2. **direct sums.**

The direct sum is equal to the weak product:

$$\bigoplus_{\alpha \in I} M_\alpha = \prod_{\alpha \in I}{}' M_\alpha$$

where the weak product is the subset of the product consisting of elements where only finitely many coordinates are nonzero:

$$\prod{}' M_\alpha = \left\{ (x_\alpha) \in \prod M_\alpha \mid x_\alpha = 0 \ \forall\forall \alpha \right\}$$

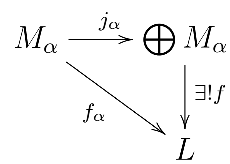
There are inclusion maps

$$j_\alpha : M_\alpha \to \bigoplus M_\alpha.$$

Every element of $\bigoplus M_\alpha$ is a sum of elements in the images of these inclusion maps.

Theorem 7.2. *The direct sum is the coproduct in the category of R-modules.*

Proof. Again, the main point is to understand the statement. The proof is trivial. This theorem says that, given any module L and homomorphisms $f_\alpha : M_\alpha \to L$, there is a unique morphism f from the direct sum to L making the following diagram commute.



The mapping f takes the element (x_α) to the sum $\sum f_\alpha(x_\alpha)$. This is defined since only finitely many of the coordinates x_α are nonzero. ☐

When the index set I is finite, the weak product is equal to the product. So, we get the following.

Corollary 7.3. *Given a finite collection of modules $M_1, \cdots, M_n$ the direct sum is equal to the direct product:*

$$\bigoplus_{i=1}^n M_i = \prod_{i=1}^n M_i.$$

The argument that proves that finite sums and products agree works in any preadditive category. Recall that a category $\mathcal{C}$ is *preadditive* if the Hom sets are additive groups and composition is biadditive.

Theorem 7.4. *If $M_1, \cdots, M_n$ are objects of any preadditive category, the finite product $\prod M_i$ exists in the category if and only if the finite sum (coproduct) $\bigoplus M_i$ exists in the category. Furthermore, they agree when they exist.*

Proof. Suppose the product $\prod M_i$ exists. Then, for each i, we have the identity map $id_{M_i} : M_i \to M_i$ and the zero maps $0 : M_i \to M_j$ when $i \neq j$. By the universal property of the product, this give a morphism

$$j_i = (0, 0, \cdots, 0, id_{M_i}, 0, \cdots, 0) : M_i \to \prod M_i$$

which has the property that

$$p_j \circ j_i = \delta_{ij}.$$

I.e., this is the identity if $i = j$ and zero if $i \neq j$. Given any object L of the category and morphisms $f_i : M_i \to L$, let $f : \prod M_i \to L$ be given by

$$f = \sum_{i=1}^{n} f_i \circ p_i.$$

Note that this uses only categorical properties: we can compose maps and we can add maps. Objects do not have elements! Since composition is biadditive we can make the following computation.

$$f \circ j_i = \sum_{j=1}^{n} f_j \circ p_j \circ j_i = \sum_{j=1}^{n} \delta_{ij} f_j = f_i$$

Thus f makes the diagram in the definition of the coproduct commute. To prove the uniqueness of f we need the following equation which holds in the endomorphism group of $\prod M_i$.

$$\sum_{i=1}^{n} j_i \circ p_i = id.$$

To prove this, let h be the endomorphism given on the left. Then

$$p_j \circ h = \sum_{i=1}^{n} p_j \circ j_i \circ p_i = \sum_{i=1}^{n} \delta_{ij} p_i = p_j.$$

By the universal property of the product, this forces $h = id$.
Returning to the uniqueness of f. Suppose that $g : \prod M_i \to L$ is another morphism so that $g \circ j_i = f_i$. Then

$$(f - g) \circ j_i = f_i - f_i = 0.$$

So,

$$0 = (f - g) \circ \sum_{i=1}^{n} j_i \circ p_i = (f - g) \circ id = f - g$$

which implies that $f = g$. □

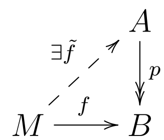
Definition 7.5. An *additive category* is a preadditive category which has finite products and sums including the empty sum which is the zero object. (Recall that a zero object is an object which is both initial and terminal.)

Thus, the corollary can be rephrased to say:

Corollary 7.6. *The category of R-modules is an additive category.*

7.3. **projective modules.**

Definition 7.7. A module M is called *projective* if, for any epimorphism $p : A \to B$ and any morphism $f : M \to B$, there is a morphism $\tilde{f} : M \to A$ so that $p \circ \tilde{f} = f$. I.e., the following diagram commutes.



(1) Free modules are projective.
(2) What happens when B is projective?
(3) Partial converse for (1).
(4) Example.

7.3.1. *free* $\Rightarrow$ *projective.*

Theorem 7.8. *Free modules are projective.*

Proof. Suppose that $M = F(X)$ is the free R-module on the set X. We have a homomorphism $f : F = M \to B$ and we want to lift it to a homomorphism $\tilde{f} : F \to A$.
We need the following adjunction property.

$$\mathrm{Hom}_R(F(X), A) \cong \mathrm{Hom}_{\mathcal{E}ns}(X, A)$$

Any homomorphism $F(X) \to A$ gives a set map $X \to A$ by restriction and any set map $g : X \to A$ extends uniquely to a homomorphism $g' : F(X) \to A$ by the formula

$$g'\left(\sum r_i x_i\right) = \sum r_i g(x_i).$$

In this case, $g : X \to A$ is given by the fact that p is surjective: For any $x_i \in X$, $f(x_i) = b_i \in B$ comes from some element $a_i \in A$. Let $g(x_i) = a_i$. Then the lifting $\tilde{f} : F \to A$ is given by

$$\tilde{f}\left(\sum r_i x_i\right) = \sum r_i g(x_i) = \sum r_i a_i.$$

This is a lifting of f since

$$f\left(\sum r_i x_i\right) = \sum r_i f(x_i) = \sum r_i b_i = p\left(\sum r_i a_i\right) = p\tilde{f}\left(\sum r_i x_i\right)$$

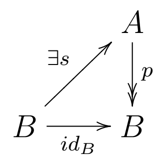
□

7.3.2. *What if B were projective?*

Theorem 7.9. *If $p : A \to B$ is onto and B is projective then*

$$A \cong B \oplus \ker p.$$

Proof. What students immediately realized is that there is a mapping $s : B \to A$. This follows from the definition of a projective module applied to the following diagram.



The morphism $s : B \to A$ is called a *section* of p because it satisfies the condition $p \circ s = id_B$. So, we have a morphism $s : B \to A$ and we also have the inclusion map $j : \ker p \to A$. Together they give a morphism

$$s \oplus j : B \oplus \ker p \to A.$$

This is the homomorphism given on elements by

$$(b, x) \mapsto s(b) + x.$$

We need to show that this is 1-1 and onto.
To show that it is onto, take any $a \in A$. Then a clearly comes from the element

$$(p(a), a - sp(a)) \in B \oplus \ker p.$$

We just have to show that $a - sp(a)$ actually lies in $\ker p$. This is a calculation:

$$p(a - sp(a)) = p(a) - pap(a) = p(a) - (id)p(a) = p(a) - p(a) = 0.$$

To show that the mapping is 1-1, suppose that $(b, x), (b', x')$ map to the same element of A. Then

$$s(b) + x = s(b') + x' \Rightarrow$$

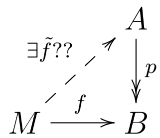
$$p(s(b) + x) = ps(b) = b = p(s(b') + x') = ps(b') = b'$$

And this implies $s(b) = s(b')$. So, $x = x'$. So, the map is a bijection. □

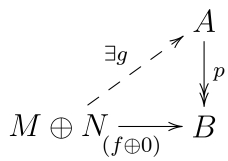
7.3.3. *partial converse.* The partial converse to the first statement is the following.

Theorem 7.10. *A module M is projective if and only if it is a direct summand of a free module.*

Proof. We know that free modules are projective. We can conclude from this that direct summands of free modules are projective. Suppose we have the diagram:



Suppose that $M \oplus N = F$ is free. Then, we get another diagram:



The lifting g exists since $M \oplus N$ is projective. But then $\tilde{f} = g|M$ is a solution of the original lifting problem. So, M is projective.
Conversely, suppose that M is projective. Then, we know that there is an epimorphism

$$p : F \to M$$

where F is free. But we just showed that, under these conditions, $F \cong M \oplus \ker p$. So, M is a direct summand of the free module F. □

7.3.4. *example.* Here is a simple example of a projective module which is not free. Let $R = M_n(K)$ be the ring of $n \times n$ matrices with coefficients in K. Then $F = R$ is a free module with action of R given by left multiplication, i.e., row operations. Matrix multiplication has the property that the entries in the different columns do not mix. (You need column operations, given by right multiplication by matrices, to do that.) Therefore, the n column vectors are direct summands. More precisely, let C_j be the set of all matrices whose entries are zero except possibly in column j. Then, we have the direct sum decomposition:

$$F = C_1 \oplus C_2 \oplus \cdots \oplus C_n$$

So, each C_j is projective (being a direct summand of the free module F). But C_j is not free since it is only n-dimensional over K and the dimension of every free module is a multiple of n^2. Note that the C_j are all isomorphic.

7.4. **finite generation.** We know that finite sums are the same as finite products. When the index set I is infinite (and the modules M_α are all nonzero), the definitions are certainly different:

$$\bigoplus M_\alpha \neq \prod M_\alpha.$$

But there is more. These are not just computationally distinct. They are conceptually different. For example, an infinite direct sum cannot be finitely generated but an infinite product can.

Theorem 7.11. *An infinite direct sum of nonzero modules cannot be finitely generated.*

Proof. Suppose that $\bigoplus M_\alpha$ has a finite set of generators $x_1, \cdots, x_n$. Then each x_i has only finitely many nonzero coordinates. Let $\beta \in I$ be a coordinate which is not one of these. Then any linear combination $\sum r_i x_i$ will also have zero as its β coordinate. So, we don't get the entire sum $\bigoplus M_\alpha$. □

Theorem 7.12. *It is possible for an infinite product of nonzero modules to be finitely generated.*

Proof. Let $R = \prod R_\alpha$ be an infinite product of rings. This is a ring. In particular it has unity: $1 = (1, 1, \cdots)$. Let $F = R$ be the free module with one generator (1). Let $M_\alpha = R_\alpha$. This is an R-module for each α. The product is

$$\prod M_\alpha = \prod R_\alpha = F$$

which is finitely generated. □

Challenge: Can you find an example where R is a reasonable ring?