MATH 101A: ALGEBRA I
PART C: TENSOR PRODUCT AND MULTILINEAR ALGEBRA

This is the title page for the notes on tensor products and multilinear algebra.

Contents

1. Bilinear forms and quadratic forms

To save time, I am talking about bilinear forms and quadratic forms at the same time. We assume throughout that R is a commutative ring. (This will make $E \otimes_R F$ into an R-module for R-modules E, F.)

1.1. definition of bilinear form.

Definition 1.1. If E, F are R-modules, a *bilinear form* on $E \times F$ is an R-bilinear map

$$f : E \times F \to R.$$

I.e.,

(1) $f(x, -) : F \to R$ is linear for each $x \in E$ and
(2) $f(-, y) : E \to R$ is linear for each $y \in F$.

In the case $E = F$, f is called a *bilinear form on E*.

Example 1.2. $E = F = R^n$ and $f : E \times F \to R$ is given by

$$f(x, y) = \sum_{i=1}^{n} x_i y_i.$$

In this example you can see why R needs to be commutative:

$$f(x, ry) = \sum x_i r y_i = r \sum x_i y_i = r f(x, y)$$

holds because $x_i r = r x_i$. This is an example of a *symmetric* bilinear form on E where symmetric means

$$f(x, y) = f(y, x).$$

Example 1.3. $E = R^n = F$ and f is given by

$$f(x, y) = \sum_{i<j} x_i y_j - x_j y_i.$$

This is an example of an *alternating form* which means

$$f(x, y) = -f(y, x).$$

Note that "symmetric" and "alternating" only apply to the case $E = F$.

Example 1.4. Take $R = \mathbb{R}$ and $E = F = C^0(I)$, the ring of continuous functions $I = [0, 1] \to \mathbb{R}$ considered as an $\mathbb{R}$-module. Then f is given by

$$f(\phi, \psi) = \int_0^1 \phi(x)\psi(x)\, dx.$$

This is a symmetric bilinear form.

The notation

$$f(x, y) = \langle x, y \rangle$$

is sometimes used, especially in this last example.

Here is a general example where $E \neq F$.

Example 1.5. Let E be any R-module and let $F = \mathrm{Hom}_R(E, R)$. Let

$$f : E \times \mathrm{Hom}_R(E, R) \to R$$

be given by

$$f(x, g) = g(x).$$

This is R-linear in x by definition and it is R-linear in g because of the way the R-module structure of $\mathrm{Hom}_R(E, R)$ is defined, namely by pointwise addition and scalar multiplication given by

$$rg(x) = g(rx)$$

This definition only works if R is commutative:

$$s(rg)(x) = rg(sx) = g(rsx) = (rs)g(x).$$

1.2. **quadratic form.** I gave two definitions of a quadratic form and showed that one implies the other.

Definition 1.6. (This definition requires $\frac{1}{2} \in R$.) A *quadratic form* on E is a function $f : E \to R$ so that

$$f(x) = \frac{1}{2} g(x, x)$$

for some symmetric bilinear form g on E.

The second definition works for any ring R.

Definition 1.7. A *quadratic form* on E is a function $f : E \to R$ so that

(1) $f(rx) = r^2 f(x)$ for every $r \in R, x \in E$.
(2) The function $g(x, y) := f(x + y) - f(x) - f(y)$ is a symmetric bilinear form on E.

To see that the second definition implies the first, suppose that $x = y$. Then

$$g(x, x) = f(2x) - 2f(x) = 4f(x) - 2f(x) = 2f(x).$$

This implies that $f(x) = \frac{1}{2} g(x, x)$ if $\frac{1}{2} \in R$. Conversely, $f(x) := \frac{1}{2} g(x, x)$ is easily seen to satisfy the second definition.

1.3. **nondegenerate bilinear forms.**

Definition 1.8. Given a bilinear form $f : E \times F \to R$ and $S \subseteq E$, I defined

$$S^\perp := \{y \in F \mid f(x, y) = 0 \ \forall x \in S\}$$

Similarly, if $T \subseteq F$ then

$$^\perp T := \{x \in E \mid f(x, y) = 0 \ \forall y \in T\}$$

Proposition 1.9. *Suppose $E = F$ and $f : E \times E \to R$ is symmetric or alternating. Then*

$$S^\perp = {}^\perp S$$

for any $S \subseteq E$.

It is easy to see that $S^\perp$ is always a submodule of F and $\perp T$ is always a submodule of E. However, in class I only pointed this out in the case of ${}^\perp F \subseteq E$ and $E^\perp \subseteq F$. These are important because they are the kernels of the maps

$$\phi_f : E \to \text{Hom}_R(F, R)$$

and

$$\psi_f : F \to \text{Hom}_R(E, R).$$

Definition 1.10. We say that $f : E \times F$ is *nondegenerate on the left* if $F^\perp = 0$, i.e., ϕ_f is a monomorphism. We can that f is *nondegenerate on the right* if $E^\perp = 0$, i.e., ψ_f is a monomorphism.

Lang uses the notation $E_0 = {}^\perp F$ and $F_0 = E^\perp$.

Proposition 1.11. *Every bilinear form $f : E \times F \to R$ induces a nondegenerate (on both sides) bilinear form*

$$\overline{f} : E/E_0 \times F/F_0 \to R$$

by the equation

$$\overline{f}(x + E_0, y + F_0) = f(x, y).$$

1.4. **nonsingular bilinear forms.**

Definition 1.12. A bilinear form $f : E \times F \to R$ is *nonsingular on the left* if

$$\phi_f : E \xrightarrow{\approx} \text{Hom}_R(F, R)$$

is an isomorphism. It is *nonsingular on the right* if

$$\psi_f : F \xrightarrow{\approx} \text{Hom}_R(E, R)$$

is an isomorphism. f is called *nonsingular* if it is nonsingular on both sides.

We discussed why nonsingularity on the left and right were different. It is because a module is not always equal to its "double dual."

Definition 1.13. The *dual* E^* of any R module E is defined by

$$E^* = \mathrm{Hom}_R(E, R).$$

If $f : E \times F \to R$ is nonsingular on the left then $\phi_f : E \cong F^*$. Nonsingularity on the right would mean that

$$\psi_f : F \cong \mathrm{Hom}_R(F^*, R) = F^{**}$$

But there are well-known examples where this is not true. There are also easy examples: If F is a torsion module over a PID R then $F^* = 0$.

The well-known example is V a countably infinite dimensional vector space over $\mathbb{Q}$. Then V^* is uncountable dimensional and V^{**} is even bigger dimensional.

There is a related isomorphism:

Proposition 1.14. *The mapping which sends a bilinear form f to $\phi_f : E \to F^*$ and to $\psi_f : F \to E^*$ gives isomorphisms of R-modules:*

$$\begin{aligned} L^2(E \times F, R) &\cong \mathrm{Hom}_R(E, \mathrm{Hom}_R(F, R)) \\ &\cong \mathrm{Hom}_R(F, \mathrm{Hom}_R(E, R)) \end{aligned}$$

Proof. The inverse of the first mapping is given by sending $\phi : E \to F^*$ to

$$f(x, y) = \phi(x)(y).$$

□

This proposition is about all possible f. For a fixed nonsingular f, we get a different formula:

Theorem 1.15. *If $f : E \times F \to R$ is nonsingular then we get an induced isomorphism of R-modules*

$$\mathrm{End}_R(E) \cong L^2(E \times F, R)$$

where $A \in \mathrm{End}_R(E)$ is mapped to the function fA given by

$$fA(x, y) = f(Ax, y).$$