MATH 101A: ALGEBRA I
PART C: TENSOR PRODUCT AND MULTILINEAR ALGEBRA

This is the title page for the notes on tensor products and multilinear algebra.

Contents

1. Bilinear forms and quadratic forms

To save time, I am talking about bilinear forms and quadratic forms at the same time. We assume throughout that R is a commutative ring. (This will make $E \otimes_R F$ into an R-module for R-modules E, F.)

1.1. definition of bilinear form.

Definition 1.1. If E, F are R-modules, a *bilinear form* on $E \times F$ is an R-bilinear map

$$f : E \times F \to R.$$

I.e.,

(1) $f(x, -) : F \to R$ is linear for each $x \in E$ and
(2) $f(-, y) : E \to R$ is linear for each $y \in F$.

In the case $E = F$, f is called a *bilinear form on* E.

Example 1.2. $E = F = R^n$ and $f : E \times F \to R$ is given by

$$f(x, y) = \sum_{i=1}^{n} x_i y_i.$$

In this example you can see why R needs to be commutative:

$$f(x, ry) = \sum x_i r y_i = r \sum x_i y_i = rf(x, y)$$

holds because $x_i r = r x_i$. This is an example of a *symmetric* bilinear form on E where symmetric means

$$f(x, y) = f(y, x).$$

Example 1.3. $E = R^n = F$ and f is given by

$$f(x, y) = \sum_{i<j} x_i y_j - x_j y_i.$$

This is an example of an *alternating form* which means

$$f(x, y) = -f(y, x).$$

Note that "symmetric" and "alternating" only apply to the case $E = F$.

Example 1.4. Take $R = \mathbb{R}$ and $E = F = C^0(I)$, the ring of continuous functions $I = [0, 1] \to \mathbb{R}$ considered as an $\mathbb{R}$-module. Then f is given by

$$f(\phi, \psi) = \int_0^1 \phi(x)\psi(x)\, dx.$$

This is a symmetric bilinear form.

The notation

$$f(x, y) = \langle x, y\rangle$$

is sometimes used, especially in this last example.

Here is a general example where $E \neq F$.

Example 1.5. Let E be any R-module and let $F = \mathrm{Hom}_R(E, R)$. Let

$$f : E \times \mathrm{Hom}_R(E, R) \to R$$

be given by

$$f(x, g) = g(x).$$

This is R-linear in x by definition and it is R-linear in g because of the way the R-module structure of $\mathrm{Hom}_R(E, R)$ is defined, namely by pointwise addition and scalar multiplication given by

$$rg(x) = g(rx)$$

This definition only works if R is commutative:

$$s(rg)(x) = rg(sx) = g(rsx) = (rs)g(x).$$

1.2. **quadratic form.** I gave two definitions of a quadratic form and showed that one implies the other.

Definition 1.6. (This definition requires $\frac{1}{2} \in R$.) A *quadratic form* on E is a function $f : E \to R$ so that

$$f(x) = \frac{1}{2} g(x, x)$$

for some symmetric bilinear form g on E.

The second definition works for any ring R.

Definition 1.7. A *quadratic form* on E is a function $f : E \to R$ so that

(1) $f(rx) = r^2 f(x)$ for every $r \in R, x \in E$.
(2) The function $g(x, y) := f(x + y) - f(x) - f(y)$ is a symmetric bilinear form on E.

To see that the second definition implies the first, suppose that $x = y$. Then

$$g(x, x) = f(2x) - 2f(x) = 4f(x) - 2f(x) = 2f(x).$$

This implies that $f(x) = \frac{1}{2} g(x, x)$ if $\frac{1}{2} \in R$. Conversely, $f(x) := \frac{1}{2} g(x, x)$ is easily seen to satisfy the second definition.

1.3. **nondegenerate bilinear forms.**

Definition 1.8. Given a bilinear form $f : E \times F \to R$ and $S \subseteq E$, I defined

$$S^\perp := \{y \in F \mid f(x, y) = 0 \ \forall x \in S\}$$

Similarly, if $T \subseteq F$ then

$$^\perp T := \{x \in E \mid f(x, y) = 0 \ \forall y \in T\}$$

Proposition 1.9. *Suppose $E = F$ and $f : E \times E \to R$ is symmetric or alternating. Then*

$$S^\perp = {}^\perp S$$

for any $S \subseteq E$.

It is easy to see that $S^\perp$ is always a submodule of F and $\perp T$ is always a submodule of E. However, in class I only pointed this out in the case of $^\perp F \subseteq E$ and $E^\perp \subseteq F$. These are important because they are the kernels of the maps

$$\phi_f : E \to \mathrm{Hom}_R(F, R)$$

and

$$\psi_f : F \to \mathrm{Hom}_R(E, R).$$

Definition 1.10. We say that $f : E \times F$ is *nondegenerate on the left* if $F^\perp = 0$, i.e., ϕ_f is a monomorphism. We can that f is *nondegenerate on the right* if $E^\perp = 0$, i.e., ψ_f is a monomorphism.

Lang uses the notation $E_0 = {}^\perp F$ and $F_0 = E^\perp$.

Proposition 1.11. *Every bilinear form $f : E \times F \to R$ induces a nondegenerate (on both sides) bilinear form*

$$\overline{f} : E/E_0 \times F/F_0 \to R$$

by the equation

$$\overline{f}(x + E_0, y + F_0) = f(x, y).$$

1.4. **nonsingular bilinear forms.**

Definition 1.12. A bilinear form $f : E \times F \to R$ is *nonsingular on the left* if

$$\phi_f : E \xrightarrow{\approx} \mathrm{Hom}_R(F, R)$$

is an isomorphism. It is *nonsingular on the right* if

$$\psi_f : F \xrightarrow{\approx} \mathrm{Hom}_R(E, R)$$

is an isomorphism. f is called *nonsingular* if it is nonsingular on both sides.

We discussed why nonsingularity on the left and right were different. It is because a module is not always equal to its "double dual."

Definition 1.13. The *dual* E^* of any R module E is defined by

$$E^* = \mathrm{Hom}_R(E, R).$$

If $f : E \times F \to R$ is nonsingular on the left then $\phi_f : E \cong F^*$. Nonsingularity on the right would mean that

$$\psi_f : F \cong \mathrm{Hom}_R(F^*, R) = F^{**}$$

But there are well-known examples where this is not true. There are also easy examples: If F is a torsion module over a PID R then $F^* = 0$.

The well-known example is V a countably infinite dimensional vector space over $\mathbb{Q}$. Then V^* is uncountable dimensional and V^{**} is even bigger dimensional.

There is a related isomorphism:

Proposition 1.14. *The mapping which sends a bilinear form f to $\phi_f : E \to F^*$ and to $\psi_f : F \to E^*$ gives isomorphisms of R-modules:*

$$\begin{aligned} L^2(E \times F, R) &\cong \mathrm{Hom}_R(E, \mathrm{Hom}_R(F, R)) \\ &\cong \mathrm{Hom}_R(F, \mathrm{Hom}_R(E, R)) \end{aligned}$$

Proof. The inverse of the first mapping is given by sending $\phi : E \to F^*$ to

$$f(x, y) = \phi(x)(y).$$

□

This proposition is about all possible f. For a fixed nonsingular f, we get a different formula:

Theorem 1.15. *If $f : E \times F \to R$ is nonsingular then we get an induced isomorphism of R-modules*

$$\mathrm{End}_R(E) \cong L^2(E \times F, R)$$

where $A \in \mathrm{End}_R(E)$ is mapped to the function fA given by

$$fA(x, y) = f(Ax, y).$$

Remark 1.16. On the second day, I pointed out that this theorem requires only that $f : E \times F \to R$ be nonsingular on the left.

Proof. We know that

$$L^2(E \times F, R) \cong \mathrm{Hom}_R(E, F^*)$$

and that a bilinear form g maps to the homomorphism $\phi_g : E \to F^*$ given by

$$\phi_g(x) = g(x, -).$$

The definition of nonsingular on the left means that $\phi_f : E \to F^*$ is an isomorphism. But, $F^* \cong E$ implies that

$$\mathrm{Hom}_R(E, F^*) \cong \mathrm{Hom}_R(E, E) = \mathrm{End}_R(E).$$

We conclude that

$$L^2(E \times F, R) \cong \mathrm{Hom}_R(E, F^*) \cong \mathrm{Hom}_R(E, E) = \mathrm{End}_R(E).$$

The only question is: What is the formula for this isomorphism?

The isomorphism is given by composition with the inverse $\phi_f^{-1} : F^* \to E$. So, g corresponds to $A = \phi_f^{-1} \circ \phi_g$ which, when applied to $x \in E$ gives

$$Ax = \phi_f^{-1}(\phi_g(x)).$$

Applying ϕ_f to both sides we got:

$$f(Ax, -) = \phi_g(x) = g(x, -)$$

which is the formula in the theorem. □

The significance of this theorem is that it shows the need for a second bilinear form, namely f, in order to express g in matrix form.

2. Matrix of a bilinear form

Suppose that $E = R^n$ and $F = R^m$ are free R-modules with bases $\{a_1, \cdots, a_n\}$ and $\{b_1, \cdots, b_m\}$. If $f : E \times F \to R$ is any bilinear form, we get a bunch of scalars:

$$f(a_i, b_j) = g_{ij} \in R.$$

These scalars determine the bilinear form f uniquely since arbitrary elements $x \in E, y \in F$ are given by

$$x = \sum x_i a_i, \qquad y = \sum y_j b_j.$$

Then

$$f(x, y) = f\left(\sum x_i a_i, \sum y_j b_j\right) = \sum_i \sum_j x_i y_j f(a_i, b_j) = \sum_i \sum_j x_i y_j g_{ij}$$

This can be written in matrix form as

$$f(X, Y) = {}^t X G Y$$

where X, Y are column vectors with coordinates x_i, y_j, ${}^t X$ is the transpose and $G = (g_{ij})$ is the $n \times m$ matrix with ij-entry equal to g_{ij}.

2.1. **matrix as homomorphism.** At this point I tried to explain that there are hidden nonsingular forms in the matrix equation. This is because the product of a row matrix and a column matrix is the their dot product:

$$f(X, Y) = \langle X, GY \rangle_E$$

where

$$\langle X, Z \rangle_E = \sum x_i z_i$$

is the dot product in E. This is a nonsingular symmetric bilinear form. The bilinear form can also be written in terms of the dot product in F:

$$f(X, Y) = \left\langle {}^t G X, Y \right\rangle_F$$

The point is that, given a fixed nonsingular form on E or F we can express bilinear forms as homomorphisms:

Proposition 2.1. *If E, F are free and finitely generated over R then a bilinear form f on $E \times F$ is given by*

$$f(X, Y) = \langle X, GY \rangle_E$$

where $G = (g_{ij}) \in \mathrm{Hom}_R(F, E)$ is uniquely determined by f.

This is just a rewording of what I explained earlier (that the matrix determines f). I just needed to explain how matrices give homomorphisms. This is just standard linear algebra but over arbitrary commutative rings.

2.2. **dual bases.** First, we need the dual basis on F. If $b_1, \cdots, b_m$ is the basis for F then we have the dual basis $b_1^*, \cdots, b_m^* \in F^*$ given by

$$b_j(x) = b_j^* \left(\sum x_i b_i \right) = x_j$$

I.e., b_j^* picks out the jth coordinate of x.

Theorem 2.2. *The b_i^* form a basis for $F^* \cong R^m$.*

Proof. First, the b_i generate F^*. To show this let $f : F \to R$ be any homomorphism. Then f is determined by the numbers $f(b_i) = c_i \in R$ and $f = \sum c_i b_i^*$ by the following:

$$f(x) = f \left(\sum x_i b_i \right) = \sum x_i f(b_i) = \sum x_i c_i = \sum c_i b_i^*(x).$$

This shows that $f = \sum c_i b_i$. Therefore, these dual elements b_i^* generate F^* and give an epimorphism

$$\phi : R^m \twoheadrightarrow F^*.$$

Next we have to show that ϕ is 1-1, i.e., its kernel is zero. So, suppose that

$$\phi(x_1, \cdots, x_m) = \sum x_i b_i^* = 0 \in F^*.$$

If we apply this element of F^* to the basis element b_j we get

$$\sum x_i b_i^*(b_j) = \sum x_i \delta_{ij} = x_j = 0.$$

Since this is true for all j, $x = (x_1, \cdots, x_m) = (0, \cdots, 0)$. So, ϕ is an isomorphism and the b_i^* form a basis for F^*. □

2.3. **basis for homomorphisms.**

Corollary 2.3. *If $E \cong R^n$ and $F \cong R^m$ are free with bases $\{a_i\}, \{b_j\}$ then $\text{Hom}_R(F, E) \cong R^{nm}$ is free with basis the functions $a_i b_j^*$ given by*

$$a_i b_j^*(y) = y_j a_i$$

if $y = \sum y_j b_j$.

This corollary says that a homomorphism is given by a linear combination

$$g = \sum_{i,j} g_{ij} a_i b_j^*$$

and the action of g is given by

$$g(x) = g \left(\sum x_j b_j \right) = \sum_{i,j} x_j g_{ij} a_i = (a_1, \cdots, a_n) G X$$

(matrix multiplication) where $G = (g_{ij})$ and $X = (x_j)$.

I gave a (much longer than necessary) categorical proof of this corollary using the following lemma.

Lemma 2.4.

$$\mathrm{Hom}_R(M, A \oplus B) \cong \mathrm{Hom}_R(M, A) \oplus \mathrm{Hom}_R(M, B)$$

Proof. This follows from the universal property of the product

$$A \times B = A \oplus B$$

namely, a homomorphism into the product $f : M \to A \oplus B$ is given uniquely by the projections $p_1 f : M \to A$ and $p_2 f : M \to B$. So, the isomorphism in the lemma is given by

$$f \mapsto (p_1 f, p_2 f)$$

and the inverse is given by

$$(f, g) \mapsto s_1 f + s_2 g$$

where $s_1 : A \to A \oplus B$ and $s_2 : B \to A \oplus B$ are the inclusion maps. □

Proof of Corollary. Since $E \cong R^n$ we have by the lemma that

$$\mathrm{Hom}_R(F, E) \cong \mathrm{Hom}_R(F, \oplus_n R) \cong \bigoplus_n \mathrm{Hom}_R(F, R) = (F^*)^n.$$

Since the ith inclusion map $s_i : R \to E \cong R^n$ is given by

$$s_i(r) = r a_i$$

the basis elements of $\mathrm{Hom}_R(F, E)$ are

$$s_i b_j^* = a_i b_j^*.$$

□

3. Naturality and contravariant functors

Somehow, I got into a long explanation about contravariant functors and naturality.

3.1. natural transformations.

The concept of "naturality" is made precise with the definition of a natural transformation. One example I used was the commutator subgroup $G' = [G, G]$ of a group G vs. its center $Z(G)$. I claimed that G' was natural but $Z(G)$ was not natural. This comes from the fact that any homomorphism $G \to H$ sends G' into H' but does not necessarily send $Z(G)$ to $Z(H)$.

Definition 3.1. Suppose that $F, G : \mathcal{A} \to \mathcal{B}$ are functors between categories $\mathcal{A}, \mathcal{B}$. Then a *natural transformation* $\eta : F \to G$ is defined to be an operation which assigns to each object $A \in \mathcal{A}$ a morphism in $\mathcal{B}$ of the form $\eta_A : FA \to GA$ so that, for all morphisms $f : A \to B$ in $\mathcal{A}$, we have the following commuting square of morphisms in $\mathcal{B}$.

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta_B} \\ GA & \xrightarrow{Gf} & GB \end{array}$$

For example, if $\mathcal{A} = \mathcal{B} = \mathcal{G}ps$, the inclusion map is a natural transformation from G' to G. This is a natural transformation from the commutator subgroup functor to the identity functor.

Another example is the torsion submodule tM of a module M over a PID R. The following commuting diagram shows that tM is a natural submodule of M and M/tM is a natural quotient module.

$$\begin{array}{ccccc} tN & \longrightarrow & N & \longrightarrow & N/tN \\ \downarrow & & \downarrow & & \downarrow \\ tM & \longrightarrow & M & \longrightarrow & M/tM \end{array}$$

The two squares in this diagram mean we have two natural transformations

$$t \to id \to id/t$$

"Natural" is not the same as "functorial." It means more. In order to have naturality, we need two functors and a natural transformation.

The dual E^* of E is thus functorial but not natural. But it is contravariantly functorial.

3.2. contravariant functors.

Definition 3.2. If $\mathcal{A}$ and $\mathcal{B}$ are categories, a *contravariant functor*

$$F : \mathcal{A} \to \mathcal{B}$$

is a pair of mappings

(1) a mapping $F : Ob(\mathcal{A}) \to Ob(\mathcal{B})$, i.e., for each $A \in \mathcal{A}$ we have $FA \in \mathcal{B}$.
(2) for every morphism $f : A \to B$ in $\mathcal{A}$ we get a morphism

$$Ff = f^* : FB \to FA$$

satisfying the two conditions:

(a) $Fid_A = (id_A)^* = id_{FA}$
(b) $(f \circ g)^* = g^* \circ f^*$.

Example 3.3. Suppose that X is any fixed R-module. Then we have a contravariant functor

$$\mathrm{Hom}_R(-, X) : R\text{-Mod} \to \mathbb{Z}\text{-Mod}$$

sending M to $\mathrm{Hom}_R(M, X)$.

Example 3.4. If X is a topological space, let $C^0(X)$ be the ring of all continuous functions $X \to \mathbb{R}$. This is a contravariant functor

$$C^0 : Top \to Rings$$

from the category of topological spaces and continuous maps to the category of rings and ring homomorphisms.

Duality $E \mapsto E^* = \mathrm{Hom}_R(E, R)$ is a contravariant functor.

So, E^* is functorial and its elements behave in a functorial way. However, the dual basis is not natural.

3.3. unnaturality of the dual.

Theorem 3.5. *There is no natural isomorphism $E \cong E^*$ in the category of finitely generated free R-modules and isomorphism.*

First, what does this theorem say? We take the category $\mathcal{C}$ whose objects are f.g. free R-modules and whose morphisms are isomorphisms. By a natural isomorphism $\phi_E : E \to E^*$ we mean an isomorphism which makes the following diagram commute for any isomorphism $f : E \cong F$ of f.g. free R modules:

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ \downarrow{\scriptstyle \phi_E} & & \downarrow{\scriptstyle \phi_F} \\ E^* & \xleftarrow[f^*]{} & F^* \end{array}$$

Proof. I proved this by contradiction. But the proof assumes the $1+1 \neq 0$, i.e., the characteristic of the ring is not 2. Suppose there is such a natural isomorphism. Let $E = F \cong R^2$ be free on two generators v, w. Then E^* is free on the dual basis elements v^*, w^*. Let $f : E \to E$ be the isomorphism given by $f(v) = v$ and $f(w) = v \pm w$ (doing two cases at once). Then $f^*(w^*) = \pm w^*$ since

$$f^*w^*(xv + yw) = w^*f(xv + yw) = w^*(xv + yv \pm yw) = \pm y$$

and $f^*(v^*) = v^* + w^*$ since

$$f^*v^*(xv + yw) = v^*f(xv + yw) = v^*(xv + yv \pm yw) = x + y.$$

Now suppose that there is some isomorphism $\phi : E \to E^*$ which makes the diagram commute. Then $\phi = f^*\phi f$. Suppose that $\phi(v) = xv^* + yw^*$. Then

$$\phi(v) = f^*\phi f(v) = f^*\phi(v) = f^*(xv^* + yw^*) = xv^* + xw^* \pm yw^*.$$

In order for this to be equal to $f(v) = xv^* + yw^*$ we must have

$$y = x + y = x - y$$

since the same isomorphism ϕ must work for both functions f. This implies that $x = 0$ and $2y = 0$. But, for ϕ to be an isomorphism, we must have y invertible in the ring. So, $2 = 0$. □

When you looked at this counterexample, you can see one of the reasons that the dual basis is not natural: When we changed the second basis element from w to $v + w$, it was the dual basis element v^* which changed. This dual element v^* depends not on v but on your choice of the other basis element w (or, in higher dimensions, v_i^* depends on all v_j where $j \neq i$). The reason is that the other basis elements must span the kernel of v_i^* by definition. So, they are the ones that determine what v_i^* are up to a scalar multiple.