4. Tensor product

Here is an outline of what I did:

1. Categorical definition
2. Construction
3. List of basic properties
4. Distributive property
5. Right exactness
6. Localization is flat
7. Extension of scalars
8. Applications

4.1. Definition. First I gave the categorical definition and then I gave an explicit construction.

4.1.1. Universal condition. Tensor product is usually defined by the following universal condition.

**Definition 4.1.** If $E, F$ are two modules over a commutative ring $R$, their tensor product $E \otimes F$ is defined to be the $R$-module having the following universal property. First, there exists an $R$-bilinear mapping

$$f : E \times F \to E \otimes F.$$  

Second, this mapping is universal in the sense that, for any other $R$-module $M$ and bilinear mapping $g : E \times F \to M$, there exists a unique $R$-module homomorphism $h : E \otimes F \to M$ making the following diagram commute.

$$
\begin{array}{ccc}
E \times F & \xrightarrow{f} & E \otimes F \\
\downarrow{g} & & \downarrow{\exists h} \\
M
\end{array}
$$

As with all universal conditions, this definition only gives the uniqueness of $E \otimes F$ up to isomorphism. For the existence we need a construction.

4.1.2. Construction of $E \otimes F$. The mapping $f : E \times F \to E \otimes F$ is not onto. However, the image must generate $E \otimes F$ otherwise we get a contradiction. The elements in the image of $f$ are denoted

$$f(x, y) = x \otimes y.$$  

**Definition 4.2.** The tensor product $E \otimes F$ is defined to be the $R$-module which is generated by the symbols $x \otimes y$ for all $x \in E, y \in F$ modulo the following conditions
(1) $x \otimes -$ is $R$-bilinear. I.e.,
   
   (a) $x \otimes ry = r(x \otimes y)$ for all $r \in R$
   
   (b) $x \otimes (y + z) = (x \otimes y) + (x \otimes z)$

(2) $- \otimes y$ is $R$-bilinear. I.e.,

   (a) $rx \otimes y = r(x \otimes y)$ for all $r \in R$

   (b) $(x + y) \otimes z = (x \otimes z) + (y \otimes z)$

I pointed out that these conditions require $R$ to be commutative since

$$rs(x \otimes y) = r(sx \otimes y) = sx \otimes ry = s(x \otimes ry) = sr(x \otimes y).$$

**Proposition 4.3.** $E \otimes F$ as given in the second definition satisfies
the universal condition of the first definitions and therefore, the tensor product exists and is unique up to isomorphism.

**Proof.** I said in class that this is obvious. If there is a bilinear mapping $g : E \times F \to M$, the induced mapping $h : E \otimes F \to M$ must take the generators $x \otimes y$ to $g(x, y)$. Otherwise the diagram will not commute. Therefore, $h$ is given on the generators and is thus unique. The only thing we need is to show that $h$ is a homomorphism. But this is equivalent to showing that the elements of the form

$$rx \otimes y - r(x \otimes y)$$

and elements corresponding to the other three conditions in the second definition go to zero in $M$. But this element goes to

$$g(rx, y) - rg(x, y) = 0$$

since $g$ is $R$-bilinear and similarly for the other three elements. So, $h$ is an $R$-module homomorphism and we are done. □

4.1.3. **functorial properties of tensor product.** The first properties I mentioned were the categorical properties which follow directly from the definition.

**Proposition 4.4.** For a fixed $R$-module $M$, tensor product with $M$ is a functor

$$M \otimes - : R-\text{Mod} \to R-\text{Mod}.$$ 

What this means is that, given an homomorphism $f : A \to B$ there is an $R$-module homomorphism

$$1 \otimes f : M \otimes A \to M \otimes B$$

which satisfies two conditions:

(1) $1 \otimes id_A = id_{M \otimes A}$
(2) $1 \otimes fg = (1 \otimes f)(1 \otimes g)$.

The definition is $(1 \otimes f)(x \otimes y) = x \otimes f(y)$. This gives a homomorphism since the mapping $M \times A \to M \otimes B$ given by

$$(x, y) \mapsto x \otimes f(y)$$

is bilinear and therefore induces the desired mapping $1 \otimes f$.

More generally, given two homomorphisms $f : M \to N, g : A \to B$ we get a homomorphism

$$f \otimes g : M \otimes A \to N \otimes B$$

by the formula

$$(f \otimes g)(x \otimes y) = f(x) \otimes g(y).$$

### 4.2. exact functors and flat modules.

Flat modules are those for which the functor $M \otimes -$ is exact. An exact functor is one that takes short exact sequences to short exact sequences. So, first I explained the definitions.

**Definition 4.5.** An *exact sequence* is a sequence of modules and homomorphisms so that the image of each map is equal to the kernel of the next map. A *short exact sequence* is an exact sequence of the following form:

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0.$$ 

In other words, $\alpha : A \to B$ is a monomorphism, $\beta : B \to C$ is an epimorphism and $\text{im} \alpha = \ker \beta$ or: $C \cong B/\alpha A$.

Sometimes short exact sequences are written:

$$A \hookrightarrow B \twoheadrightarrow C.$$ 

**Definition 4.6.** A functor $F : R-\text{Mod} \to R-\text{Mod}$ is called *exact* if it takes short exact sequences to short exact sequences. Thus the short exact sequence above should give the short exact sequence

$$0 \to FA \xrightarrow{F\alpha} FB \xrightarrow{F\beta} FC \to 0.$$ 

**Definition 4.7.** An $R$-module $M$ is called *flat* if $M \otimes -$ is an exact functor. I.e.,

$$0 \to M \otimes A \xrightarrow{1 \otimes \alpha} M \otimes B \xrightarrow{1 \otimes \beta} M \otimes C \to 0$$

is exact for all short exact sequences $A \hookrightarrow B \twoheadrightarrow C$.

One of the main results (which we will see is actually trivial) is that $S^{-1}R$ is flat for any multiplicative set $S$. I.e., localization is exact.
4.3. **list of properties.** I explained that the exactness of localization was one of the key ideas. However, the explanation required an understanding of the basic properties of tensor product. So, I went back to the beginning with this list.

- (0) (unity) \( R \otimes M \cong M \).
- (1) (commutative) \( M \otimes N \cong N \otimes M \).
- (2) (distributive) \( N \otimes \oplus M_i \cong \bigoplus (N \otimes M_i) \).
- (3) (associative) \( (A \otimes B) \otimes C \cong A \otimes (B \otimes C) \).
- (4) (right exactness) \( M \otimes - \) is right exact, i.e., a short exact sequence \( A \rightarrow B \rightarrow C \) gives an exact sequence
  \[
  M \otimes A \xrightarrow{1 \otimes \alpha} M \otimes B \xrightarrow{1 \otimes \beta} M \otimes C \rightarrow 0
  \]
- (5) (localization is exact) I.e., we get an exact sequence:
  \[
  0 \rightarrow S^{-1}A \rightarrow S^{-1}B \rightarrow S^{-1}C \rightarrow 0.
  \]
- (6) (extension of scalars) Given a ring homomorphism \( R \rightarrow S \), every \( R \)-module \( M \) gives an \( S \) module \( S \otimes_R M \).

4.3.1. **Grothendieck ring.** I did not prove properties (1) and (3). I said they were obvious. However, I put the first three conditions into a conceptual framework by pointing out that these are the axioms of a ring. The only thing that we don’t have is an additive inverse. The algebraic construction is as follows.

First, you take the set of all isomorphism classes of finitely generated \( R \)-modules \([M]\). This set has addition and multiplication given by

\[
[M] + [N] = [M \oplus N] \\
[M][N] = [M \otimes N]
\]

Addition and multiplication are associative and commutative and have units: \([0]\) is the additive unit and \([R]\) is the multiplicative unit. It just doesn’t have additive inverses. So, Grothendieck said to just put in formal inverses:

\[
[M] - [N]
\]

which are defined like fractions:

\[
[M] - [N] = [A] - [B]
\]

if there exists another module \( C \) so that

\[
M \oplus B \oplus C \cong N \oplus A \oplus C.
\]

This gives a ring whose name is \( G(R) \). The notation \( K_0(R) \) is for the ring of formal differences of f.g. projective \( R \)-modules.
4.3.2. $R \otimes M \cong M$. After using this formula many times in the lecture, I decided I should prove it. I put the proof at the beginning in the notes where it belongs.

**Theorem 4.8.** $R \otimes M \cong M$ for any $R$-module $M$.

**Proof.** Since the mapping

$$R \times M \to M$$

given by $(r, x) \mapsto rx$ is bilinear it induces a mapping

$$\mu : R \otimes M \to M$$

so that $\mu(r \otimes x) = rx$. The inverse mapping $\phi : M \to R \otimes M$ is given by $\phi(x) = 1 \otimes x$. We carefully checked that these are inverse to each other:

$$\phi \mu(r \otimes x) = \phi(rx) = 1 \otimes rx = r(1 \otimes x) = r \otimes x$$
$$\mu \phi(x) = \mu(1 \otimes x) = 1x = x.$$

So, these maps are both isomorphisms of $R$-modules. \hfill \square

4.3.3. **distributive property.** I gave a category theory proof of the distributivity of tensor product over direct sum. First I pointed out that the following formal characterization of direct sum.

**Lemma 4.9.** $M$ is the direct sum of modules $M_1, \cdots, M_n$ if and only if there are inclusion maps $s_i : M_i \to M$ and projection maps $p_i : M \to M_i$ so that

1. $p_j \circ s_i = \delta_{ij}$, i.e., equal to the identity mapping on $M_i$ if $i = j$

2. $\sum_{i=1}^n s_i \circ p_i = id_M$.

I drew the following diagrams to illustrate these equations.

$$
\xymatrix{ M_i \ar[rr]^\delta_{ij} \ar[dr]_{s_i} & & M_j \ar[dl]_{p_j} \\
& M & }
$$

$$
\xymatrix{ M \ar[rr] \ar[dr]_{p_i} & & M \\
& M \ar[ur]_{s_i} & }
$$

This lemma was proved in any preadditive category in Part B, Theorem 7.4.
Theorem 4.10. If \( M \cong \bigoplus_{i=1}^{n} M_i \) then
\[
N \otimes M = N \otimes \bigoplus_{i=1}^{n} M_i \cong \bigoplus_{i=1}^{n} (N \otimes M_i).
\]

Proof. Consider the homomorphisms:
\[
N \otimes M_i \xrightarrow{1 \otimes s_i} N \otimes M \xrightarrow{1 \otimes p_j} N \otimes M_j
\]

a) \((1 \otimes p_j)(1 \otimes s_i) = 1 \otimes p_j s_i = 1 \otimes \delta_{ij} = \delta_{ij}(1 \otimes 1)\).
b) \(\sum(1 \otimes s_i)(1 \otimes p_i) = 1 \otimes \sum s_i p_i = 1 \otimes 1 = id_{N \otimes M}\).

These conditions imply that \( N \otimes M \cong \bigoplus N \otimes M_i \) by the above lemma.

Remark 4.11. This proof works in any preadditive category to show that any linear functor distributes over direct sum.

4.4. right exactness of tensor product. I didn’t prove the right exactness of tensor product this first time because the elementary proof is messy and not very instructive. I just explained that this is a special case of a much more general principle that: “All linear left adjoint functors are right exact.” I will explain this later. The statement of the theorem is the following.

Theorem 4.12. Tensor product with any \( R \)-module \( M \) sends any exact sequence of \( R \)-modules of the form:
\[
A \overset{\alpha}{\rightarrow} B \overset{\beta}{\rightarrow} C \rightarrow 0
\]
to another exact sequence of the same form:
\[
M \otimes A \rightarrow M \otimes B \rightarrow M \otimes C \rightarrow 0.
\]

This statement appears stronger than the original statement since the hypothesis is weaker. But I explained that the first statement implies this second version. Suppose that we know that \( M \otimes - \) sends short exact sequences to right exact sequences as above. Then how can we conclude that it sends the more general right exact sequences \( A \rightarrow B \rightarrow C \rightarrow 0 \) to right exact sequences?

The first statement implies that \( M \otimes - \) takes epimorphisms to epimorphisms. (In fact this is obvious since the generators \( x \otimes y \in M \otimes C \) come from generators \( x \otimes \tilde{y} \in M \otimes B \).) Therefore \( M \otimes A \) maps onto \( M \otimes \alpha(A) \). If we assume the weaker condition that the functor \( M \otimes - \)
takes short exact sequences to right exact sequences, then it will take the short exact sequence

$$0 \rightarrow \alpha(A) \hookrightarrow B \xrightarrow{\beta} C \rightarrow 0$$

to an exact sequence

$$M \otimes \alpha(A) \rightarrow M \otimes B \xrightarrow{1 \otimes \beta} M \otimes C \rightarrow 0$$

This says that $M \otimes \alpha(A)$ maps onto the kernel of $1 \otimes \beta$. But $M \otimes A$ maps onto $M \otimes \alpha(A)$. So, $M \otimes A$ also maps onto $\text{ker}(1 \otimes \beta)$. So, we get an exact sequence

$$M \otimes A \xrightarrow{1 \otimes \alpha} M \otimes B \xrightarrow{1 \otimes \beta} M \otimes C \rightarrow 0.$$ 

Here is an example of how this is used.

**Corollary 4.13.** Suppose that $I \subset R$ is an ideal. Then

$$R/I \otimes M \cong M/IM$$

where $IM$ is the submodule of $M$ generated by all products of the form $ax$ where $a \in I$ and $x \in M$. In particular, when $I = (p)$ is principal, we have

$$R/(p) \otimes M \cong M/pM$$

where $pM = \{px \mid x \in M\}$.

*Proof.* Suppose that $I$ is generated by elements $a_i$. Then we have an epimorphism of $R$ modules

$$
\bigoplus_i R \twoheadrightarrow I
$$

sending $(r_i) \in \bigoplus R$ to $\sum r_ia_i \in I$. This gives an exact sequence

$$
\bigoplus_i R \xrightarrow{\alpha} R \twoheadrightarrow R/I \rightarrow 0.
$$

Tensor with $M$ to give

$$
\bigoplus_i R \otimes M \xrightarrow{\alpha \otimes 1} R \otimes M \rightarrow R/I \otimes M \rightarrow 0.
$$

Using the isomorphisms $\mu : R \otimes M \cong M$ and $\phi : M \cong R \otimes M$ we get an exact sequence

$$
\bigoplus_i M \xrightarrow{\mu(\alpha \otimes 1) \phi} M \rightarrow R/I \otimes M \rightarrow 0
$$

where $\mu(\alpha \otimes 1) \phi$ sends $(x_i) \in \bigoplus M$ to $\sum a_ix_i \in M$. The image is equal to $IM$ by definition. So, $R/I \otimes M \cong M/IM$ as claimed. \qed
For finitely generated modules over a PID we can now compute the tensor product:

\[ M \otimes \left( R^n \oplus \bigoplus R/(p^n_i) \right) \cong M^n \oplus \bigoplus M/p^n_i M. \]

4.5. localization is exact. Recall that a multiplicative set is a subset \( S \subseteq R \) which is closed under multiplication, contains 1 and does not contain 0. The localization \( S^{-1}R \) was defined to be the ring of all fractions \( r/s \) where \( r \in R \) and \( s \in S \) modulo the equivalence relation

\[ \frac{r}{s} \sim \frac{r'}{s'} \]

if there is an element \( t \in S \) so that \( rs't = r'st \). This ring is also an \( R \)-module since we have an action of \( R \) given by

\[ r \cdot \frac{x}{s} = \frac{rx}{s}. \]

**Proposition 4.14.** For any \( R \)-module \( M \) let \( S^{-1}M \) be the set of equivalence classes of fractions \( x/s \) where \( x \in M, s \in S \) modulo the equivalence relation \( x/s \sim y/s' \) if there is a \( t \in S \) so that \( ts'x = tsy \). Then \( S^{-1}M \) is an \( R \)-module with action of \( R \) given by \( r(x/s) = rx/s \) and

\[ S^{-1}M \cong S^{-1}R \otimes M. \]

**Proof.** There is an obvious map \( S^{-1}R \otimes M \to S^{-1}M \) sending \( r/s \otimes x \) to \( rx/s \). The inverse map sends \( x/s \) to \( 1/s \otimes x \). To show that this is well-defined, take an equivalent element \( tx/ts \). This goes to

\[ \frac{1}{ts} \otimes tx = t \left( \frac{1}{ts} \otimes x \right) = \frac{t}{ts} \otimes x = \frac{1}{s} \otimes x. \]

The rest of the proof is straightforward. \( \square \)

**Theorem 4.15.** \( S^{-1}R \) is a flat \( R \)-module. Equivalently, every short exact sequence of \( R \)-modules \( A \to B \to C \) induces an exact sequence

\[ 0 \to S^{-1}A \to S^{-1}B \to S^{-1}C \to 0. \]

**Proof.** Since tensor product is right exact, it suffices to show that \( S^{-1}A \to S^{-1}B \) is a monomorphism. This is easy. We can assume that \( A \subseteq B \) and suppose that \( a \in A \) and \( s \in S \) so that the element \( a/s \in S^{-1}A \) goes to zero in \( S^{-1}B \). This means

\[ \frac{a}{s} \sim \frac{0}{s} \]

since
in $S^{-1}B$. By definition this is equivalent to saying that there exists $t \in S$ so that $tsa = 0$. But this same equation implies that $a/s = 0/1$ in $S^{-1}A$. So, we are done.

The ring $S^{-1}R$ acts on the module $S^{-1}M$ in the obvious way:

\[
\frac{r}{s} \frac{x}{t} = \frac{rx}{st}.
\]

This makes $S^{-1}M$ into a module over $S^{-1}R$. This is an example of “extension of scalars.”

4.6. extension of scalars. We had a concept before called “restriction of scalars.” That was when we had a subring $S$ of $R$ or, more generally, a ring homomorphism $\phi : S \to R$ and we got an induced map

\[\phi^* : R\text{-Mod} \to S\text{-Mod}\]

which sent an $R$-module $M$ to the same thing with the action of $S$ given by $s \cdot x = \phi(s)x$. I.e., we restricted the action of the ring to $S$.

“Extension of scalars” goes the other way.

**Proposition 4.16.** Given a ring homomorphism $\phi : R \to S$ and an $R$-module $M$, $S \otimes_R M$ is an $S$-module with action of $S$ given by

\[s(t \otimes x) = st \otimes x.\]

The module is sometimes written as $S \otimes_\phi M$ because the $R$-module structure is given by

\[r(s \otimes x) = (\phi(r)s) \otimes x = s \otimes rx.\]

This is the $R$-module structure induced from the $S$-module structure by restriction of scalars.

**Proof.** Multiplication by elements of $S$ gives an $R$-linear map $S \to S$ and therefore gives an $R$-linear map $S \otimes M \to S \otimes M$ by naturality of tensor product. This gives a sequence of ring homomorphisms

\[S \to \mathrm{End}_R(S) \to \mathrm{End}_R(S \otimes M)\]

which defines the $S$-module structure on $S \otimes M$. □

One special case of this is when $R$ is a domain and $F = Q(R)$ is the field of fractions.

**Definition 4.17.** Suppose that $M$ is a module over a domain $R$. Then the rank of $M$ is defined to be the dimension of $Q(R) \otimes M$ as a vector space over the field $Q(R)$.

\[r(M) = \dim_{Q(R)} Q(R) \otimes M.\]
Theorem 4.18. For a f.g. module $M$ over a PID $R$, if

$$M \cong R^r \oplus \bigoplus \frac{R}{(p_i^{n_i})},$$

the number $r$ is equal to the rank of $M$ and is therefore uniquely determined.

Proof. This is a calculation using the fact that

$$R/(a) \otimes Q(R) \cong Q(R)/aQ(R) = 0$$

since $aQ(R) = Q(R)$ for $a \neq 0$:

$$Q(R) \otimes M = Q(R) \otimes R^r \oplus \bigoplus \frac{Q(R)}{p_i^{n_i}Q(R)} \cong Q(R)^r.$$

It still remains to show that the numbers $p_i^{n_i}$ are uniquely determined.