4. Tensor product

Here is an outline of what I did:

1. categorical definition
2. construction
3. list of basic properties
4. distributive property
5. right exactness
6. localization is flat
7. extension of scalars
8. applications

4.1. **definition.**

First I gave the categorical definition and then I gave an explicit construction.

4.1.1. *universal condition.* Tensor product is usually defined by the following universal condition.

Definition 4.1. If E, F are two modules over a commutative ring R, their *tensor product* $E \otimes F$ is defined to be the R-module having the following universal property. First, there exists an R-bilinear mapping

$$f : E \times F \to E \otimes F.$$

Second, this mapping is universal in the sense that, for any other R-module M and bilinear mapping $g : E \times F \to M$, there exists a unique R-module homomorphism $h : E \otimes F \to M$ making the following diagram commute.

$$\begin{tikzcd}
E \times F \arrow[rr, "f"] \arrow[dr, "g"'] & & E \otimes F \arrow[dl, dashed, "\exists! h"] \\
& M &
\end{tikzcd}$$

As with all universal conditions, this definition only gives the uniqueness of $E \otimes F$ up to isomorphism. For the existence we need a construction.

4.1.2. *construction of* $E \otimes F$. The mapping $f : E \times F \to E \otimes F$ is not onto. However, the image must generate $E \otimes F$ otherwise we get a contradiction. The elements in the image of f are denoted

$$f(x, y) = x \otimes y.$$

Definition 4.2. The tensor product $E \otimes F$ is defined to be the R module which is generated by the symbols $x \otimes y$ for all $x \in E, y \in F$ modulo the following conditions

1. $x \otimes -$ is R-bilinear. I.e.
    1. $x \otimes ry = r(x \otimes y)$ for all $r \in R$
    2. $x \otimes (y + z) = (x \otimes y) + (x \otimes z)$
2. $- \otimes y$ is R-bilinear. I.e.,
    1. $rx \otimes y = r(x \otimes y)$ for all $r \in R$
    2. $(x + y) \otimes z = (x \otimes z) + (y \otimes z)$

I pointed out that these conditions require R to be commutative since

$$rs(x \otimes y) = r(sx \otimes y) = sx \otimes ry = s(x \otimes ry) = sr(x \otimes y).$$

Proposition 4.3. *$E \otimes F$ as given in the second definition satisfies the universal condition of the first definitions and therefore, the tensor product exists and is unique up to isomorphism.*

Proof. I said in class that this is obvious. If there is a bilinear mapping $g : E \times F \to M$, the induced mapping $h : E \otimes F \to M$ must take the generators $x \otimes y$ to $g(x, y)$. Otherwise the diagram will not commute. Therefore, h is given on the generators and is thus unique. The only thing we need is to show that h is a homomorphism. But this is equivalent to showing that the elements of the form

$$rx \otimes y - r(x \otimes y)$$

and elements corresponding to the other three conditions in the second definition go to zero in M. But this element goes to

$$g(rx, y) - rg(x, y) = 0$$

since g is R-bilinear and similarly for the other three elements. So, h is an R-module homomorphism and we are done. □

4.1.3. *functorial properties of tensor product.* The first properties I mentioned were the categorical properties which follow directly from the definition.

Proposition 4.4. *For a fixed R-module M, tensor product with M is a functor*

$$M \otimes - : R\text{-}Mod \to R\text{-}Mod.$$

What this means is that, given an homomorphism $f : A \to B$ there is an R-module homomorphism

$$1 \otimes f : M \otimes A \to M \otimes B$$

which satisfies two conditions:

1. $1 \otimes id_A = id_{M \otimes A}$

(2) $1 \otimes fg = (1 \otimes f)(1 \otimes g)$.

The definition is $(1\otimes f)(x\otimes y) = x\otimes f(y)$. This gives a homomorphism since the mapping $M \times A \to M \otimes B$ given by

$$(x, y) \mapsto x \otimes f(y)$$

is bilinear and therefore induces the desired mapping $1 \otimes f$.

More generally, given two homomorphisms $f : M \to N, g : A \to B$ we get a homomorphism

$$f \otimes g : M \otimes A \to N \otimes B$$

by the formula

$$(f \otimes g)(x \otimes y) = f(x) \otimes g(y).$$

4.2. **exact functors and flat modules.** Flat modules are those for which the functor $M \otimes -$ is exact. An exact functor is one that takes short exact sequences to short exact sequences. So, first I explained the definitions.

Definition 4.5. An *exact sequence* is a sequence of modules and homomorphisms so that the image of each map is equal to the kernel of the next map. A *short exact sequence* is an exact sequence of the following form:

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0.$$

In other words, $\alpha : A \to B$ is a monomorphism, $\beta : B \to C$ is an epimorphism and $\operatorname{im}\alpha = \ker\beta$ or: $C \cong B/\alpha A$.

Sometimes short exact sequences are written:

$$A \rightarrowtail B \twoheadrightarrow C.$$

Definition 4.6. A functor $F : R\text{-Mod} \to R\text{-Mod}$ is called *exact* if it takes short exact sequences to short exact sequences. Thus the short exact sequence above should give the short exact sequence

$$0 \to FA \xrightarrow{F\alpha} FB \xrightarrow{F\beta} FC \to 0.$$

Definition 4.7. An R-module M is called *flat* if $M \otimes -$ is an exact functor. I.e.,

$$0 \to M \otimes A \xrightarrow{1\otimes\alpha} M \otimes B \xrightarrow{1\otimes\beta} M \otimes C \to 0$$

is exact for all short exact sequences $A \rightarrowtail B \twoheadrightarrow C$.

One of the main results (which we will see is actually trivial) is that $S^{-1}R$ is flat for any multiplicative set S. I.e., localization is exact.

4.3. **list of properties.** I explained that the exactness of localization was one of the key ideas. However, the explanation required an understanding of the basic properties of tensor product. So, I went back to the beginning with this list.

(0) (unity) $R \otimes M \cong M$.
(1) (commutative) $M \otimes N \cong N \otimes M$
(2) (distributive) $N \otimes \oplus M_i \cong \bigoplus (N \otimes M_i)$
(3) (associative) $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$
(4) (right exactness) $M \otimes -$ is right exact, i.e., a short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ gives an exact sequence

$$M \otimes A \xrightarrow{1\otimes\alpha} M \otimes B \xrightarrow{1\otimes\beta} M \otimes C \to 0$$

(5) (localization is exact) I.e., we get an exact sequence:

$$0 \to S^{-1}A \to S^{-1}B \to S^{-1}C \to 0.$$

(6) (extension of scalars) Given a ring homomorphism $R \to S$, every R-module M gives an S module $S \otimes_R M$.

4.3.1. *Grothedieck ring.* I did not prove properties (1) and (3). I said they were obvious. However, I put the first three conditions into a conceptual framework by pointing out that these are the axioms of a ring. The only thing that we don't have is an additive inverse. The algebraic construction is as follows.

First, you take the set of all isomorphism classes of finnitely generated R-modules $[M]$. This set has addition and multiplication given by

$$[M] + [N] = [M \oplus N]$$
$$[M][N] = [M \otimes N]$$

Addition and multiplication are associative and commutative and have units: $[0]$ is the additive unit and $[R]$ is the multiplicative unit. It just doesn't have additive inverses. So, Grothendieck said to just put in formal inverses:

$$[M] - [N]$$

which are defined like fractions:

$$[M] - [N] = [A] - [B]$$

if there exists another module C so that

$$M \oplus B \oplus C \cong N \oplus A \oplus C.$$

This gives a ring whose name is $G(R)$. The notation $K_0(R)$ is for the ring of formal differences of f.g. projective R-modules.

4.3.2. $R\otimes M \cong M$. After using this formula many times in the lecture, I decided I should prove it. I put the proof at the beginning in the notes where it belongs.

Theorem 4.8. *$R \otimes M \cong M$ for any R-module M.*

Proof. Since the mapping

$$R \times M \to M$$

given by $(r, x) \mapsto rx$ is bilinear it induces a mapping

$$\mu : R \otimes M \to M$$

so that $\mu(r \otimes x) = rx$. The inverse mapping $\phi : M \to R \otimes M$ is given by $\phi(x) = 1 \otimes x$. We carefully checked that these are inverse to each other:

$$\phi\mu(r \otimes x) = \phi(rx) = 1 \otimes rx = r(1 \otimes x) = r \otimes x$$

$$\mu\phi(x) = \mu(1 \otimes x) = 1x = x.$$

So, these maps are both isomorphisms of R-modules. □

4.3.3. *distributive property.* I gave a category theory proof of the distributivity of tensor product over direct sum. First I pointed out that the following formal characterization of direct sum.

Lemma 4.9. *M is the direct sum of modules $M_1, \cdots, M_n$ if and only if there are inclusion maps $s_i : M_i \to M$ and projection maps $p_i : M \to M_i$ so that*

1. *$p_j \circ s_i = \delta_{ij}$, i.e., equal to the identity mapping on M_i if $i = j$ and equal to 0 if $i \neq j$.*
2. *$\sum s_i \circ p_i = id_M$.*

I drew the following diagrams to illustrate these equations.

$$\begin{tikzcd}
M_i \arrow[rr, "\delta_{ij}"] \arrow[dr, "s_i"'] & & M_j \\
& M \arrow[ur, "p_j"'] &
\end{tikzcd}
\qquad p_j \circ s_i = \delta_{ij}$$

$$\begin{tikzcd}
M \arrow[rr] \arrow[dr, "p_i"'] & & M \\
& M \arrow[ur, "s_i"'] &
\end{tikzcd}
\qquad \textstyle\sum_{i=1}^{n} s_i \circ p_i = id_M$$

This lemma was proved in any preadditive category in Part B, Theorem 7.4.

Theorem 4.10. *If* $M \cong \bigoplus_{i=1}^n M_i$ *then*

$$N \otimes M = N \otimes \bigoplus_{i=1}^n M_i \cong \bigoplus_{i=1}^n (N \otimes M_i)\,.$$

Proof. Consider the homomorphisms:

$$N \otimes M_i \xrightarrow{1\otimes s_i} N \otimes M \xrightarrow{1\otimes p_j} N \otimes M_j$$

a) $(1\otimes p_j)(1\otimes s_i) = 1\otimes p_j s_i = 1\otimes \delta_{ij} = \delta_{ij}(1\otimes 1)$.
b) $\sum(1\otimes s_i)(1\otimes p_i) = 1\otimes \sum s_i p_i = 1\otimes 1 = id_{N\otimes M}$.

These conditions imply that $N\otimes M \cong \bigoplus N\otimes M_i$ by the above lemma. □

Remark 4.11. This proof works in any preadditive category to show that any linear functor distributes over direct sum.

4.4. **right exactness of tensor product.**

I didn't prove the right exactness of tensor product this first time because the elementary proof is messy and not very instructive. I just explained that this is a special case of a much more general principle that: "All linear left adjoint functors are right exact." I will explain this later. The statement of the theorem is the following.

Theorem 4.12. *Tensor product with any R-module M sends any exact sequence of R-modules of the form:*

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

to another exact sequence of the same form:

$$M\otimes A \to M\otimes B \to M\otimes C \to 0.$$

This statement appears stronger than the original statement since the hypothesis is weaker. But I explained that the first statement implies this second version. Suppose that we know that $M\otimes -$ sends short exact sequences to right exact sequences as above. Then how can we conclude that it sends the more general right exact sequences $A \to B \to C \to 0$ to right exact sequences?

The first statement implies that $M\otimes -$ takes epimorphisms to epimorphisms. (In fact this is obvious since the generators $x\otimes y \in M\otimes C$ come from generators $x\otimes \tilde{y} \in M\otimes B$.) Therefore $M\otimes A$ maps onto $M\otimes \alpha(A)$. If we assume the weaker condition that the functor $M\otimes -$

takes short exact sequences to right exact sequences, then it will take the short exact sequence

$$0 \to \alpha(A) \hookrightarrow B \xrightarrow{\beta} C \to 0$$

to an exact sequence

$$M \otimes \alpha(A) \to M \otimes B \xrightarrow{1\otimes\beta} M \otimes C \to 0$$

This says that $M \otimes \alpha(A)$ maps onto the kernel of $1 \otimes \beta$. But $M \otimes A$ maps onto $M \otimes \alpha(A)$. So, $M \otimes A$ also maps onto $\ker(1 \otimes \beta)$. So, we get an exact sequence

$$M \otimes A \xrightarrow{1\otimes\alpha} M \otimes B \xrightarrow{1\otimes\beta} M \otimes C \to 0.$$

Here is an example of how this is used.

Corollary 4.13. *Suppose that $I \subset R$ is an ideal. Then*

$$R/I \otimes M \cong M/IM$$

where IM is the submodule of M generated by all products of the form ax where $a \in I$ and $x \in M$. In particular, when $I = (p)$ is principal, we have

$$R/(p) \otimes M \cong M/pM$$

where $pM = \{px \mid x \in M\}$.

Proof. Suppose that I is generated by elements a_i. Then we have an epimorphism of R modules

$$\bigoplus_i R \twoheadrightarrow I$$

sending $(r_i) \in \bigoplus R$ to $\sum r_i a_i \in I$. This gives an exact sequence

$$\bigoplus_i R \xrightarrow{\alpha} R \to R/I \to 0.$$

Tensor with M to give

$$\bigoplus_i R \otimes M \xrightarrow{\alpha\otimes 1} R \otimes M \to R/I \otimes M \to 0.$$

Using the isomorphisms $\mu : R \otimes M \cong M$ and $\phi : M \cong R \otimes M$ we get an exact sequence

$$\bigoplus_i M \xrightarrow{\mu(\alpha\otimes 1)\phi} M \to R/I \otimes M \to 0$$

where $\mu(\alpha \otimes 1)\phi$ sends $(x_i) \in \bigoplus M$ to $\sum a_i x_i \in M$. The image is equal to IM by definition. So, $R/I \otimes M \cong M/IM$ as claimed. □

For finitely generated modules over a PID we can now compute the tensor product:

$$M \otimes \left(R^n \oplus \bigoplus R/(p_i^{n_i})\right) \cong M^n \oplus \bigoplus M/p_i^{n_i}M.$$

4.5. **localization is exact.** Recall that a *multiplicative set* is a subset $S \subseteq R$ which is closed under multiplication, contains 1 and does not contain 0. The *localization* $S^{-1}R$ was defined to be the ring of all fractions r/s where $r \in R$ and $s \in S$ modulo the equivalence relation

$$\frac{r}{s} \sim \frac{r'}{s'}$$

if there is an element $t \in S$ so that $rs't = r'st$. This ring is also an R-module since we have an action of R given by

$$r \cdot \frac{x}{s} = \frac{rx}{s}.$$

Proposition 4.14. *For any R-module M let $S^{-1}M$ be the set of equivalence classes of fractions x/s where $x \in M, s \in S$ modulo the equivalence relation $x/s \sim y/s'$ if there is a $t \in S$ so that $ts'x = tsy$. Then $S^{-1}M$ is an R-module with action of R given by $r(x/s) = rx/s$ and*

$$S^{-1}M \cong S^{-1}R \otimes M.$$

Proof. There is an obvious map $S^{-1}R \otimes M \to S^{-1}M$ sending $r/s \otimes x$ to rx/s. The inverse map sends x/s to $1/s \otimes x$. To show that this is well-defined, take an equivalent element tx/ts. This goes to

$$\frac{1}{ts} \otimes tx = t\left(\frac{1}{ts} \otimes x\right) = \frac{t}{ts} \otimes x = \frac{1}{s} \otimes x.$$

The rest of the proof is straightforward. □

Theorem 4.15. *$S^{-1}R$ is a flat R-module. Equivalently, every short exact sequence of R-modules $A \rightarrowtail B \twoheadrightarrow C$ induces an exact sequence*

$$0 \to S^{-1}A \to S^{-1}B \to S^{-1}C \to 0.$$

Proof. Since tensor product is right exact, it suffices to show that $S^{-1}A \to S^{-1}B$ is a monomorphism. This is easy. We can assume that $A \subseteq B$ and suppose that $a \in A$ and $s \in S$ so that the element $a/s \in S^{-1}A$ goes to zero in $S^{-1}B$. This means

$$\frac{a}{s} \sim \frac{0}{s}$$

in $S^{-1}B$. By definition this is equivalent to saying that there exists $t \in S$ so that $tsa = 0$. But this same equation implies that $a/s = 0/1$ in $S^{-1}A$. So, we are done. $\square$

The ring $S^{-1}R$ acts on the module $S^{-1}M$ in the obvious way:

$$\frac{r}{s}\frac{x}{t} = \frac{rx}{st}.$$

This makes $S^{-1}M$ into a module over $S^{-1}R$. This is an example of "extension of scalars."

4.6. **extension of scalars.** We had a concept before called "restriction of scalars." That was when we had a subring S of R or, more generally, a ring homomorphism $\phi : S \to R$ and we got an induced map

$$\phi^* : R\text{-}\operatorname{Mod} \to S\text{-}\operatorname{Mod}$$

which sent an R-module M to the same thing with the action of S given by $s \cdot x = \phi(s)x$. I.e., we restricted the action of the ring to S.

"Extension of scalars" goes the other way.

Proposition 4.16. *Given a ring homomorphism $\phi : R \to S$ and an R-module M, $S \otimes_R M$ is an S-module with action of S given by*

$$s(t \otimes x) = st \otimes x.$$

The module is sometimes written as $S \otimes_\phi M$ because the R-module structure is given by

$$r(s \otimes x) = (\phi(r)s) \otimes x = s \otimes rx.$$

This is the R-module structure induced from the S-module structure by restriction of scalars.

Proof. Multiplication by elements of S gives an R-linear map $S \to S$ and therefore gives an R-linear map $S \otimes M \to S \otimes M$ by naturality of tensor product. This gives a sequence of ring homomorphisms

$$S \to \operatorname{End}_R(S) \to \operatorname{End}_R(S \otimes M)$$

which defines the S-module structure on $S \otimes M$. $\square$

One special case of this is when R is a domain and $F = Q(R)$ is the field of fractions.

Definition 4.17. Suppose that M is a module over a domain R. Then the *rank* of M is defined to be the dimension of $Q(R) \otimes M$ as a vector space over the field $Q(R)$.

$$r(M) = \dim_{Q(R)} Q(R) \otimes M.$$

Theorem 4.18. *For a f.g. module M over a PID R, if*

$$M \cong R^r \oplus \bigoplus R/(p_i^{n_i}),$$

the number r is equal to the rank of M and is therefore uniquely determined.

Proof. This is a calculation using the fact that

$$R/(a) \otimes Q(R) \cong Q(R)/aQ(R) = 0$$

since $aQ(R) = Q(R)$ for $a \neq 0$:

$$Q(R) \otimes M = Q(R) \otimes R^r \oplus \bigoplus Q(R)/p_i^{n_i}Q(R) \cong Q(R)^r.$$

□

It still remains to show that the numbers $p_i^{n_i}$ are uniquely determined.