10 Semidirect Products

A semidirect product is a groups having two complementary subgroups one of which is normal.

**Definition 10.1.** Two subgroups $H, K \leq G$ are called *complementary* (to each other) if

1. $H \cap K = 1$
2. $HK = G$ [where $HK = \{hk | h \in H, k \in K\}$]

For example, if $G$ is a group of order $p^aq^b$ then the $p$-Sylow subgroup $P$ and $q$-Sylow subgroup $Q$ are complementary. The complement of $H$ is not unique, even if both $H, K$ are normal. [E.g., take $G = \mathbb{R}^2$ as an additive group and let $H, K$ be any two distinct lines through the origin.]

**Proposition 10.2.** If $H, K \leq G$ are complementary then each element of $G$ can be written uniquely in the form

$$g = hk$$

where $h \in H, k \in K$.

*Proof.* We are assuming that $G = HK$ so each $g = hk$. So we only need the uniqueness. But $g = h_1k_1 = h_2k_2$ implies $h_2^{-1}h_1 = k_2k_1^{-1}$. This is in $H \cap K = 1$ so $h_1 = h_2$ and $k_1 = k_2$. \qed

**Proposition 10.3.** If $K \leq G$ then any complementary subgroup $H$ is isomorphic to $Q = G/K$. Furthermore the isomorphism $q : H \rightarrow Q$ is the restriction to $H$ of the quotient map $G \twoheadrightarrow Q$.

*Proof.* The kernel of the map $\phi : H \rightarrow Q$ induced by the quotient map is $K \cap H = 1$. Thus $\phi$ is injective. To show that $\phi$ is onto take any element $gK$ of $Q = G/K$. Then $g = hk$ so $gK = hkK = hK = \phi(h) \in \text{im}(\phi)$. \qed

**Definition 10.4.** $G$ is a *semidirect product* of $K$ by $Q$ if $K \leq G$ and there exists $H \leq G$ is a complement of $K$ isomorphic to $Q$.

By Proposition 10.3, $Q \cong H \cong G/K$. Thus a semidirect product of $K$ by $Q$ is an extension of $K$ by $Q$.

**Example 10.5.** $S_n$ is a semidirect product of $A_n$ by $\mathbb{Z}/2$ where the complement of $A_n$ is given, e.g., by $H = \langle (12) \rangle$.

**Example 10.6.** The dihedral group $D_{2n}$ is a semidirect product of $\mathbb{Z}/n = \langle s \rangle$ by $\mathbb{Z}/2$ where $H = \langle t \rangle$ is the complement of $\langle s \rangle$.

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1Note that, in general, $HK$ is not a subgroup of $G$ unless either $H$ or $K$ is normal. For example $H = \langle (12) \rangle$, $K = \langle (234) \rangle$ generate all of $S_4$ but $HK$ has only 6 elements.
Example 10.7. If \( n \) is odd, the cyclic group \( \mathbb{Z}/2n \) is also a semidirect product of \( \mathbb{Z}/n \) by \( \mathbb{Z}/2 \) where the complement of \( K = 2\mathbb{Z}/2n \) is the 2-Sylow subgroup of \( \mathbb{Z}/2n \).

HW4.ex05: Prove that \( Q \) (the quaternion group of order 8) is not a semidirect product of \( \mathbb{Z}/4 \) by \( \mathbb{Z}/2 \).

Theorem 10.8. \( G \) is a semidirect product of \( K \) by \( Q = G/K \) iff the quotient map \( q : G \to Q \) splits, i.e., if there exists a homomorphism \( s : Q \to G \) so that \( qs = id_Q \). [\( s \) is called a section of \( q \).]

Proof. The image of the section \( s \), if it exists, is a complement for \( K \) so \( G \) is a semidirect product. Conversely, if \( G \) is a semidirect product, then \( K \) has a complement \( H \) and the inverse of the isomorphism \( \phi : H \to Q \) is a section \( s \). \( \square \)

Definition 10.9. Suppose that \( \theta : Q \to \text{Aut}(K) \) \((x \mapsto \theta_x)\) is a homomorphism. Then we say that a semidirect product \( G \) of \( K \) by \( Q \) realizes \( \theta \) if the conjugation action of \( H = s(Q) \) on \( K \) is given by \( \theta \), i.e.,

\[
\gamma_{s(x)} = \theta_x
\]

for all \( x \in Q \).

The basic theorem about semidirect products is that they exist and are uniquely determined (up to isomorphism) by \( K, Q, \theta \). I will outline the usual proof and then explain a fancy categorical proof (using holomorphs and pullbacks) which although more complicated has a certain appeal.

Definition 10.10. Given \( K, Q \) and \( \theta : Q \to \text{Aut}(K) \) let \( K \rtimes_\theta Q \) be the cartesian product \( K \times Q \) with the following multiplication rule:

\[
(a, x)(b, y) = (a\theta_x(b), xy).
\]

We need to check that this is a group operation.

1. (associativity) Use the fact that \( \theta \) is a homomorphism \((\theta_{xy} = \theta_x\theta_y)\):

   (a) \(((a, x)(b, y))(c, z) = (a\theta_x(b), xy)(c, z) = (a\theta_x(b)\theta_{xy}(c), xyz)\)

   (b) \((a, x)((b, y)(c, z)) = (a, x)(b\theta_y(c), yz) = (a\theta_x(b)\theta_x\theta_y(c), xyz)\)

2. (left identity) \((1, 1)(a, x) = (1\theta_1(a), x) = (a, x)\)

3. (left inverse) \((\theta_z(a)^{-1}, z)(a, x) = (\theta_z(a)^{-1}\theta_z(a), zx) = (1, 1) \) if \( z = x^{-1} \).

Theorem 10.11. \( K \rtimes_\theta Q \) is a semidirect product of \( K \) by \( Q \) realizing \( \theta \).
Proof. We identify $K$ with the set of all $(a, 1)$ where $a \in K$. This is a normal subgroup of $K \rtimes \theta Q$:

$$(*, x)(c, 1)(*, x^{-1}) = (*\theta_x(c)\ast, xx^{-1}) \in K$$

$H = \{(1, x) \mid x \in Q\}$ is a subgroup complementary to $K$ and isomorphic to $Q$. Thus $K \rtimes \theta Q$ is a semidirect product of $K$ by $Q$. It is also clear that this realizes the action $\theta$. [Replace $\ast$ by $1$ in the calculation above.]

**Theorem 10.12.** Any semidirect product $G$ of $K$ by $Q$ realizing $\theta$ is isomorphic to $K \rtimes \theta Q$.

**Proof.** By definition 10.9 there is a section $s : Q \to G$ of the quotient map $q : G \to Q$ so that $\theta_x = \gamma_s(x)$. Let $\psi : K \rtimes \theta Q \to G$ be given by $\psi(a, x) = as(x)$. This is a homomorphism since

$$\psi(a, x)\psi(b, y) = as(x)bs(y) = a\gamma_s(x)(b)s(x)s(y) = a\theta_x(b)s(xy) = \psi(a\theta_x(b), xy)$$

and $\psi$ is a bijection by Proposition 10.2. \hfill \Box

There is a “universal” semidirect product of $K$ called the “holomorph” of $K$.

**Definition 10.13.** For any group $K$ its *holomorph* $\text{Hol}(K)$ is defined to be the group of permutations of $K$ generated by

1. automorphisms $\psi : K \to K$
2. left translations $L_g : K \to K$ ($x \mapsto gx$)

In other words,

$$\text{Hol}(K) = \langle \text{Aut}(K), K^\ell \rangle$$

where $K^\ell = \{L_g \mid g \in K\}$.

Note that $\text{Aut}(K)$ normalizes $K^\ell$ since

$$\psi L_g \psi^{-1} = L_{\psi(g)}$$

Consequently, $K^\ell \leq \text{Hol}(K)$ and $\text{Hol}(K) = K^\ell \text{Aut}(K)$.

**Theorem 10.14.** $\text{Hol}(K) \cong K \rtimes id \text{Aut}(K)$ where $id : \text{Aut}(K) \to \text{Aut}(K)$ is the identity homomorphism.

**Proof.** The elements of $\text{Hol}(K)$ can be written uniquely in the form $L_g \psi$ and multiplication is given by:

$$L_g \psi L_h \phi = L_g L_{\psi(h)} \psi \phi = L_{g \psi(h)} \psi \phi$$

which corresponds to the multiplication rule in $K \rtimes id \text{Aut}(K)$. \hfill \Box

The semidirect product $K \rtimes \theta Q$ is now given by the “pull-back” construction:

$$K \rtimes \theta Q = \{(x, f) \mid x \in Q, f = L_g \psi \in \text{Hol}(K), \theta(x) = \psi\}$$