6 Solvable groups

Definition 6.1. A group $G$ is solvable if it has a subnormal series

$$G = G_0 \geq G_1 \geq G_2 \geq \cdots \geq G_n = 1$$

where each quotient $G_i/G_{i+1}$ is an abelian group. We will call this a solvable series.

For example, any abelian group is solvable even if it is infinite. Another more interesting example is the symmetric group $S_4$ which has the solvable series:

$$S_4 \triangleright A_4 \triangleright K \triangleright 1$$

with quotients $S_4/A_4 \cong \mathbb{Z}/2$, $A_4/K \cong \mathbb{Z}/3$ and $K/1 = K \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ where $K$ is the Klein 4-group given by:

$$K = \{1, (12)(34), (13)(24), (14)(23)\}$$

Since these are all of the elements of $S_4$ of cycle type $(ab)(cd)$, $K$ is normal in $S_4$.

Some other easy theorems:

Theorem 6.2. Every finite $p$-group is solvable.

Proof. We already proved that every finite $p$-group has a normal series where each quotient group is cyclic of order $p$. □

Theorem 6.3. If $H \triangleleft G$ and both $H$ and $G/H$ are solvable then $G$ is solvable.

Proof. This is obvious. Let $\phi : G \rightarrow Q = G/H$ be the quotient map. Then a solvable series for $G$ is given by:

$$G = \phi^{-1}(Q_0) \geq \phi^{-1}(Q_1) \geq \cdots \geq \phi^{-1}(Q_n) = H = H_0 \geq H_1 \geq \cdots \geq H_m = 1$$

where $Q = Q_0 \geq \cdots \geq Q_n = 1$ and $H = H_0 \geq \cdots \geq H_m = 1$ are solvable series for $Q, H$. □

Corollary 6.4. If $G$ and $H$ are solvable then so is $G \times H$.

If $G$ is a solvable group it has a canonical subnormal series with abelian quotients. This is the derived series of $G$:

$$G \geq G' \geq G'' \geq G''' \geq \cdots$$

Recall that the commutator subgroup $G' = [G, G]$ is the subgroup of $G$ generated by all commutators $[g, h] = ghg^{-1}h^{-1} = \phi_g(h)h^{-1}$ and $G'' = (G')'$, $G''' = ((G')')'$, etc. We use the shorthand $G^{(n)}$ when this is done $n$ times so, e.g., $G^{(4)} = G^{''''}$. We recall some basic properties of commutator subgroups.
Proposition 6.5. For any group $G$ we have:

1. $G' \trianglelefteq G$.\(^1\)
2. $G/G'$ is abelian. More generally,
3. $G/H$ is abelian iff $H \supseteq G'$.\(^2\)
4. $H' \leq G'$ if $H \leq G$.\(^3\)

We will see later that the derived series is a normal series. However, it is obviously subnormal by 6.5.1 and we have:

Theorem 6.6. $G$ is solvable if and only if $G^{(n)} = 1$ for some $n$.

Proof. It is enough to show that each $G_k$ in a solvable series of $G$ contains $G^{(k)}$. The first step: $G' \leq G_1$ follows from 6.5.3 above. If $G^{(k)} \leq G_k$ then $G^{(k+1)} = (G^{(k)})' \leq (G_k)' \leq G_{k+1}$ by 6.5.3 since $G_k/G_{k+1}$ is abelian. \(\blacksquare\)

Theorem 6.7. Every subgroup of a solvable group is solvable.

Proof. By Prop. 6.5.4 we have: $H^{(n)} \leq G^{(n)} \leq 1$. \(\blacksquare\)

Lemma 6.8. For any homomorphism $f : G \to H$ we have $f(G') = f(G)' \leq H'$. In particular, $\phi(G') = G'$ for any automorphism $\phi$ of $G$.

Proof. $f$ sends the generators of $G'$ to the generators of $f(G)'$ since
\[
f([a, b]) = f(aba^{-1}b^{-1}) = f(a)f(b)f(a)^{-1}f(b)^{-1} = [f(a), f(b)]
\]

\(\blacksquare\)

Theorem 6.9. Every quotient group of a solvable group is solvable.

Proof. Let $f : G \to Q$ be an epimorphism. Then $Q^{(n)} = f(G^{(n)}) = 1$. \(\blacksquare\)

Definition 6.10. A subgroup $H$ of $G$ is called characteristic if $\phi(H) = H$ for every automorphism $\phi$ of $G$. We write $H \text{ char } G$.

Examples of characteristic subgroups:

1. $G'$ char $G$
2. $Z(G)$ char $G$
3. $\Phi(G)$ char $G$ [\(\Phi(G)\), the Frattini subgroup of $G$ is the intersection of all maximal proper subgroups of $G$. (\(\Phi(G) = G\) is there are none.)]
4. $nA$ char $A$ if $A$ is an additive group.

\(^1\)A conjugate of a commutator is a commutator: $[a, b]^c = [a^c, b^c]$ where $x^c = c^{-1}xc$.

\(^2\)\(G/H\) is abelian iff for each $a, b$ we have $Hab = Hba \leftrightarrow H[a, b] = H \leftrightarrow [a, b] \subseteq H$.

\(^3\)The generators $[h_1, h_2]$ of $H'$ form a subset of the set of generators of $G'$. 
These are all obviously characteristic since they are intrinsically defined. Another not so obvious example is: $A_n$ char $S_n$. This is because $A_n$ is generated by all elements of order 3 in $S_n$. This is an intrinsic description of $A_n$ since automorphisms take elements of order 3 to elements of order 3.

**Lemma 6.11.**  
1. $H$ char $G$ implies $H \trianglelefteq G$
2. $K$ char $H$ char $G$ implies $K$ char $G$
3. $K$ char $N \trianglelefteq G$ implies $K \trianglelefteq G$

HW3.ex 03: Give a counterexample to the statement: $N \trianglelefteq K$ char $G \Rightarrow N \trianglelefteq G$.

**Proof.**  
1. Characteristic subgroups of $G$ are invariant under all automorphisms of $G$. Normal subgroups are only required to be invariant under inner automorphisms.
3. Any inner automorphism of $G$ leaves $N$ invariant and the induced automorphism of $N$ leaves $K$ invariant.

**Theorem 6.12.** The higher commutators $G^{(k)}$ are characteristic subgroups of $G$. In particular, if $G$ is solvable, the derived series is solvable series which is also a normal series.

Instead of going from top to bottom we can find a solvable series for a solvable group from bottom to top using the following theorem.

**Theorem 6.13.** Any minimal normal subgroup of a solvable group is elementary abelian.

**Proof.** Suppose that $N \trianglelefteq G$ is minimal normal. Then $N$ is solvable by 6.7. Consequently, $N' < N$. But $N'$ char $N \Rightarrow N' \trianglelefteq G$ so $N' = 1$, i.e., $N$ is abelian. If $p$ divides $|N|$ then $pN$ is a characteristic proper subgroup of $N$ so $pN \trianglelefteq G$ and must be trivial. Thus $N$ is an elementary abelian $p$-group.

HW3.ex 04: If $S, T$ are solvable subgroups of $G$ and $S \trianglelefteq G$ then show that $ST$ is solvable.