20 Torsion groups

Recall:

- \( tG = \{x | nx = 0 \text{ for some } n > 0 \} \) is the torsion subgroup of \( G \).
- \( 0 \to \frac{tG}{\text{torsion}} \to G \to \frac{G/tG}{\text{torsion-free}} \to 0 \)
- This sequence does not always split (i.e., \( tG \) is not necessarily a direct summand of \( G \)), e.g., \( G = \prod_{p \text{ prime}} \mathbb{Z}/p \).

Now let \( G \) be any torsion group. Then we will show that there exists a short exact sequence

\[ 0 \to \bigoplus \text{cyclic} \to G_{\text{torsion}} \to D_{\text{divisible}} \to 0, \]

i.e., every torsion group is an extension of a direct sum of cyclic groups by a divisible group.

Recall: Divisible \( D = \bigoplus \mathbb{Q} \oplus \bigoplus \mathbb{Z}/p^\infty \)

In order to prove the above theorem, we need the notions of pure and basic subgroups.

**Definition 20.1.** \( S \leq G \) is pure if \( ng \cap S = nS \), (i.e., if \( x \in S \) and \( x = ng \) then \( \exists s \in S \ni x = ns \)).

**Example 20.2.** \( S = 2\mathbb{Z}/8 \) is not pure in \( G = \mathbb{Z}/8 \) but \( S = 2\mathbb{Z}/6 \) is pure in \( G = \mathbb{Z}/6 \) since \( \mathbb{Z}/6 \cong \mathbb{Z}/3 \oplus \mathbb{Z}/2 \).

**Facts:**

1. Every direct summand of \( G \) is pure in \( G \).
2. The converse is not true. (We will discuss when it is.)
3. If \( G/S \) is torsion-free then \( S \leq G \) is pure.
4. \( tG \) is pure in \( G \) (but not necessarily a direct summand).
5. \( S \leq T \) pure, \( T \leq G \) pure \( \Rightarrow \) \( S \leq G \) pure.
6. \( S \leq T \leq G \) with \( S \leq G \) pure and \( T/S \leq G/S \) pure \( \Rightarrow \) \( T \leq G \) pure.\(^1\)
7. \( S \leq G \) pure, \( \overline{y} \in G/S \). Then \( \exists y \in G \ni \overline{y} = y + S \) and \( \text{order}(y) = \text{order}(\overline{y}) \).\(^2\)

\(^1\)If \( t = ng \) then \( t + S = nx + S \) for some \( x \in T \) since \( T/S \) is pure in \( G/S \). Consequently, \( n(g - x) \in S \) so there is an \( s \in S \) so that \( n(g - x) = ns \). But then \( t = n(x + s) \in nT \).

\(^2\)If \( z \) is a representative of \( \overline{y} \) and \( n = \text{order}(\overline{y}) \) then \( n\overline{y} = nz + S = S \) so \( nz \in S \). Since \( S \) is pure \( \exists s \in S \ni nz = ns \). But then \( z - s \) is a representative of \( \overline{y} \) of order \( n \).
8. \( S_1 \leq S_2 \leq \cdots \leq G \), \( S_i \leq G \) pure \( \Rightarrow \cup S_i \leq G \) pure.

Recall:

- \( G \) torsion \( \Rightarrow \bigoplus G_p \), \( G_p \) \( p \)-primary groups.
- If \( g \in G \) has order \( m \) and \((n, m) = 1\) then \( g \) is divisible by \( n \). \((\alpha m + \beta n = 1 \Rightarrow g = \alpha m g + \beta n g = n \beta g)\)
- \( G[p] = \{ x \in G \mid px = 0 \} \).

**Lemma 20.3.** Let \( G \) be \( p \)-primary and not divisible. Then \( G \) had a pure cyclic subgroup.

**Proof.** Suppose first that \( G \neq pG \) then \( \exists y \in G \) which is not divisible by \( p \).

Claim \( \langle y \rangle \leq G \) is pure cyclic (and \( p \)-primary).

Proof: \( \langle y \rangle \) is clearly cyclic so let \( ay \in \langle y \rangle \) and suppose \( ay = ng \). (WTS: \( ay = n(\langle y \rangle) \)) If \((p, n) = 1 \) then \( n \) divides all elements of \( \langle y \rangle \) as recalled above. Thus suppose \( n = p \) and \( ay = pg \).

Case 1. \( a = pa' \). Then \( ay = p(a'y) \).

Case 2. \((a, p) = 1 \). Then \( \alpha a + \beta p = 1 \Rightarrow y = \alpha ay + \beta py = \alpha pg + \beta py = p(\alpha g + \beta y) \).

Now suppose \( G = pG \). Then \( G \) is divisible since every element is divisible by \( p \) and we already know that \( p \)-primary elements are divisible by all \( q \) relatively prime to \( p \). \( \square \)

In order to extend this to a direct sum of cyclic groups, we need pure independent sets.

Recall: \( X \subset G \) is independent if \( (\sum m_\alpha x_\alpha = 0 \Rightarrow m_\alpha x_\alpha = 0) \) or equivalently, \( \langle X \rangle = \bigoplus_{x_\alpha \in X} \langle x_\alpha \rangle \).

**Definition 20.4.** \( X \subset G \) is pure independent if \( X \) is independent and \( \langle X \rangle \leq G \) is pure.

**Lemma 20.5.** If \( G \) is \( p \)-primary and \( X \subset G \) is maximal pure independent then \( G/\langle X \rangle \) is divisible.

**Proof.** Suppose not, i.e., suppose \( G/\langle X \rangle \) is not divisible. Then, by Lemma 20.3, there is a pure cyclic subgroup \( \langle \overline{y} \rangle \leq G/\langle X \rangle \). By Fact 6, \( \exists \overline{y} \in G \) s.t. \( \text{order}(\overline{y}) = \text{order}(\overline{y}) \) and \( y + \langle X \rangle = \overline{y} \).

Claim \( X \cup \{y\} \) is pure independent (contradicting the maximality of \( X \)).

\( \langle X \cup \{y\} \rangle \) pure: \( \langle X \rangle \) is pure in \( G \) and \( \langle \overline{y} \rangle = \langle X \cup \{y\} \rangle / \langle X \rangle \) is pure in \( G/\langle X \rangle \). By Fact 6 this implies \( \langle X \cup \{y\} \rangle \) is pure in \( G \).

\( X \cup \{y\} \) independent: Suppose \( \sum m_\alpha x_\alpha + m_y y = 0 \). Then \( m_y y = 0 \Rightarrow m_y y = 0 \Rightarrow \sum m_\alpha x_\alpha = 0 \Rightarrow \) each \( m_\alpha x_\alpha = 0 \) since \( X \) is independent. \( \square \)

**Definition 20.6.** \( B \leq G \) is a basic subgroup if

1. \( B \) is a direct sum of cyclic groups,
2. $B \leq G$ is pure and
3. $G/B$ is divisible.

**Theorem 20.7 (Kulikov).** Every torsion group $G$ has a basic subgroup, i.e., $G$ is an extension of a direct sum of cyclic groups by a divisible group:

$$0 \to \bigoplus \text{cyclic} \xrightarrow{\text{pure}} G \to D_{\text{divisible}} \to 0$$

**Proof.** $G$ torsion $\Rightarrow$ $G = \bigoplus G_p$, $G_p$ $p$-primary. If $B_p \leq G_p$ is basic then $\bigoplus B_p$ is basic in $G = \bigoplus G_p$. Thus we may assume that $G$ is $p$-primary.

If $G$ is divisible let $B = 0$.

If $G$ is not divisible then $G$ contains a pure cyclic subgroup $\langle y \rangle$. Thus $\{y\}$ is a pure independent subset of $G$. By Zorn’s lemma there is a maximal pure independent set $X$. (Given any tower $X_\alpha$ of pure independent sets, $\bigcup X_\alpha$ is independent and $\langle \bigcup X_\alpha \rangle = \bigcup \langle X_\alpha \rangle$ is pure in $G$ by Fact 8.) By Lemma 20.5, $G/\langle X \rangle$ is divisible. By independence we also have: $\langle X \rangle = \bigoplus \langle x_\alpha \rangle$. So $\langle X \rangle$ is basic in $G$.

**Corollary 20.8.** If $G$ is a torsion group of bounded order ($nG = 0$) then $G$ is a direct sum of cyclic groups.

**Proof.** $0 \to \bigoplus \text{cyclic} \to G \to D \to 0$. $nG = 0$ $\Rightarrow$ $nD = 0$. But $D = nD$ since $D$ is divisible. Thus $D = 0$ and $G = \bigoplus \text{cyclic}$. □

Now we want to show that any two basic subgroups of a torsion group are isomorphic. It suffices to consider the $p$-primary case.

**Definition 20.9.** The Ulm invariants of a $p$-primary group $G$ are given by:

$$U(n, G) = \dim_{\mathbb{Z}/p} \frac{p^n G \cap G[p]}{p^{n+1} G \cap G[p]}$$

**Lemma 20.10.** Let $G$ be a direct sum of cyclic groups of $p$-power order. Then $U(n, G)$ is equal to the number of cyclic summands of $G$ of order $p^{n+1}$.

**Proof.** Let $G_i$ be the direct sum of all summands of $G$ of order $p^i$. Then

$$G = G_1 \oplus G_2 \oplus \cdots \oplus G_{n+1} \oplus G_{n+2} \oplus \cdots$$

$$G[p] = G_1 \oplus pG_2 \oplus \cdots \oplus p^n G_{n+1} \oplus p^{n+1} G_{n+2} \oplus \cdots$$

$$p^n G = 0 \oplus 0 \oplus \cdots \oplus p^n G_{n+1} \oplus p^n G_{n+2} \oplus \cdots$$

$$p^n G \cap G[p] = 0 \oplus 0 \oplus \cdots \oplus p^n G_{n+1} \oplus p^{n+1} G_{n+2} \oplus \cdots$$

$$p^{n+1} G \cap G[p] = 0 \oplus 0 \oplus \cdots \oplus 0 \oplus p^{n+1} G_{n+2} \oplus \cdots$$

$$\frac{p^n G \cap G[p]}{p^{n+1} G \cap G[p]} \cong p^n G_{n+1} \Rightarrow$$

$$U(n, G) = \dim_{\mathbb{Z}/p} p^n G_{n+1} = \# \text{cyclic summands of } G \text{ of order } p^{n+1}$$

□

3
Corollary 20.11. G, H p-primary direct sums of cyclic groups, then G ∼= H ⇔ U(n, G) = U(n, H) for all n ≥ 0.

Theorem 20.12. Any two basic subgroups of a p-primary group G are isomorphic.

Proof. Let B be a basic subgroup of G.

• The number of cyclic summands of order $p^n$ in B is equal to the number of cyclic summands of order $p$ in $B/p^{n+1}B$, i.e.,

$$U(n-1, B) = U(0, B/p^{n+1}B)$$

• $G = B + p^n G$

Pf: $G/B$ divisible so each $g + B = \overline{g} \in G/B$ is divisible by $p^n$, i.e.,

$$g + B = p^n x + B \Rightarrow g = b + p^n x \in B + p^n G.$$  

• $B/p^{n+1}B \cong G/p^{n+1}G$

Pf: $\frac{G}{p^{n+1}G} = \frac{B + p^{n+1}G}{p^{n+1}G} \cong \frac{B}{B/p^{n+1}G} = \frac{B}{p^{n+1}B}$ since $B$ is pure.

So $U(n - 1, B) = U(0, G/p^{n+1}G)$ depends only on $G$.  

We pointed out earlier that pure subgroups are not necessarily direct summands. Now we give a sufficient conditions when it is a direct summand and give some interesting corollaries.

Corollary 20.13 (Prüfer). A pure subgroup of bounded order is a direct summand.

Proof. Let $S \leq G$ pure with $nS = 0$. Let $\pi : G \rightarrow G/(S + nG)$ be the quotient map.

• $G/(S + nG)$ has bounded order $n$, so it is a sum of cyclic groups (20.8).

• $G/(S + nG) = \bigoplus \langle \overline{x}_\alpha \rangle$, $r_\alpha = \text{order}(\overline{x}_\alpha)$ divides $n$.

• Choose $x_\alpha \in G \ni \pi(x_\alpha) = \overline{x}_\alpha$.

• $r_\alpha x_\alpha = s_\alpha + ng_\alpha \Rightarrow s_\alpha = r_\alpha x_\alpha - ng_\alpha = r_\alpha(x_\alpha - (n/r_\alpha)g_\alpha)$.

• $S \leq G$ pure $\Rightarrow \exists s'_\alpha \ni s_\alpha = r_\alpha s'_\alpha$.

• Let $y_\alpha = x_\alpha - s'_\alpha$. Then $\pi(y_\alpha) = \pi(x_\alpha) = \overline{x}_\alpha$ so $r_\alpha y_\alpha = r_\alpha x_\alpha - r_\alpha s'_\alpha = s_\alpha + ng_\alpha - s_\alpha = ng_\alpha \in nG$.

• Let $K = \langle nG \cup \{y_\alpha\} \rangle$.  


Claim 1: $S \cap K = 0$.
\begin{align*}
Pf: \; s &= ng + \sum m_\alpha y_\alpha \Rightarrow \pi(s) = \pi(ng) + \sum m_\alpha \pi(y_\alpha) = \sum m_\alpha \pi(x_\alpha) = 0 \\
m_\alpha \pi(x_\alpha) = 0 &\Rightarrow r_\alpha | m_\alpha \Rightarrow m_\alpha y_\alpha \in nG \Rightarrow s \in nG \cap S = nS = 0.
\end{align*}

Claim 2: $S + K = G$.
\begin{align*}
Pf: \; g \in G &\Rightarrow \pi(g) = \sum _\ell \alpha x_\alpha = \sum _\ell \alpha \pi(y_\alpha) = g - \sum _\ell \alpha y_\alpha \in \ker \pi = S + nG \Rightarrow g \in s + (nG + \sum _\ell \alpha y_\alpha) \subseteq S + K.
\end{align*}
Therefore, $S \oplus K = G$.

**Corollary 20.14.** If $tG$ has bounded order then $G \cong tG \oplus G/tG$.

**Proof.** $tG \leq G$ is pure.

**Corollary 20.15.** If $G$ is torsion and not divisible then $G$ has a $p$-primary cyclic direct summand.

**Proof.** Since $G = \bigoplus G_p$ we may assume $G$ is $p$-primary. By Lemma 20.3, $G$ has a pure cyclic subgroup which must be a direct summand by Prüfer.

**Corollary 20.16.** If $G$ is indecomposable then $G$ is either torsion or torsion-free.

**Proof.** If not then we have $0 \rightarrow tG \rightarrow G \rightarrow G/tG \rightarrow 0$. If $tG$ is divisible then $G \cong tG \oplus G/tG$, a contradiction. If $tG$ is not divisible then $\exists S \leq tG$ pure and cyclic (20.3). Since $tG$ is pure in $G$, $S$ is pure in $G$ so $S$ is a direct summand of $G$. by 20.13.