

Part D
Representations of finite groups

MATH 101B: ALGEBRA II
PART D: REPRESENTATIONS OF FINITE GROUPS

For the rest of the semester we will discuss representations of groups, mostly finite groups. An elementary treatment would start with characters and their computation. But the Wedderburn structure theorem will allow us to start at a higher level which will give more meaning to the character tables which we will be constructing. This is from Lang, XVIII, §1-7 with additional material from Serre's "Linear Representations of Finite Groups" (Springer Graduate Texts in Math 42) and Alperin and Bell's "Groups and Representations" (Springer GTM 162).

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1. THE GROUP RING $k[G]$

The main idea is that representations of a group G over a field k are “the same” as modules over the group ring $k[G]$. First I defined both terms.

1.1. Representations of groups.

Definition 1.1. A representation of a group G over a field k is defined to be a group homomorphism

$$\rho : G \rightarrow \text{Aut}_k(V)$$

where V is a vector space over k .

Here $\text{Aut}_k(V)$ is the group of k -linear automorphisms of V . This also written as $GL_k(V)$. This is the group of units of the ring $\text{End}_k(V) = \text{Hom}_k(V, V)$ which, as I explained before, is a ring with addition defined pointwise and multiplication given by composition. If $\dim_k(V) = d$ then $\text{Aut}_k(V) \cong \text{Aut}_k(k^d) = GL_d(k)$ which can also be described as the group of units of the ring $\text{Mat}_d(k)$ or as:

$$GL_d(k) = \{A \in \text{Mat}_d(k) \mid \det(A) \neq 0\}$$

$d = \dim_k(V)$ is called the *dimension* of the representation ρ .

1.1.1. examples.

Example 1.2. The first example I gave was the trivial representation. This is usually defined to be the one dimensional representation $V = k$ with trivial action of the group G (which can be arbitrary). Trivial action means that $\rho(\sigma) = 1 = id_V$ for all $\sigma \in G$.

In the next example, I pointed out that the group G needs to be written multiplicatively no matter what.

Example 1.3. Let $G = \mathbb{Z}/3$. Written multiplicatively, the elements are $1, \sigma, \sigma^2$. Let $k = \mathbb{R}$ and let $V = \mathbb{R}^2$ with $\rho(\sigma)$ defined to be rotation by $120^\circ = 2\pi/3$. I.e.,

$$\rho(\sigma) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$$

Example 1.4. Suppose that E is a field extension of k and $G = \text{Gal}(E/k)$. Then G acts on E by k -linear transformations. This gives a representation:

$$\rho : G \hookrightarrow \text{Aut}_k(E)$$

Note that this map is an inclusion by definition of “Galois group.”

1.1.2. *axioms.* In an elementary discussion of group representations I would write a list of axioms as a definition. However, they are just longwinded explanations of what it means for $\rho : G \rightarrow \text{Aut}_k(V)$ to be a group homomorphism. The only advantage is that you don't need to assume that $\rho(\sigma)$ is an automorphism. Here are the axioms. (I switched the order of (2) and (3) in the lecture.)

- (1) $\rho(1) = 1$ $1v = v \quad \forall v \in V$
- (2) $\rho(\sigma\tau) = \rho(\sigma)\rho(\tau) \quad \forall \sigma, \tau \in G$ $(\sigma\tau)v = \sigma(\tau v) \quad \forall v \in V$
- (3) $\rho(\sigma)$ is k -linear $\forall \sigma \in G$ $\sigma(av + bw) = a\sigma v + b\sigma w \quad \forall v, w \in V, a, b \in k$

The first two conditions say that ρ is an *action* of G on V . Actions are usually written by juxtaposition:

$$\sigma v := \rho(\sigma)(v)$$

The third condition says that the action is k -linear. So, together, the axioms say that a representation of G is a k -linear action of G on a vector space V .

1.2. **Modules over $k[G]$.** The *group ring* $k[G]$ is defined to be the set of all finite k linear combinations of elements of G : $\sum a_\sigma \sigma$ where $a_\sigma \in k$ for all $\sigma \in G$ and $a_\sigma = 0$ for almost all σ .

For example, $\mathbb{R}[\mathbb{Z}/3]$ is the set of all linear combinations

$$x + y\sigma + z\sigma^2$$

where $x, y, z \in \mathbb{R}$. I.e., $\mathbb{R}[\mathbb{Z}/3] \cong \mathbb{R}^3$. In general $k[G]$ is a vector space over k with G as a basis.

Multiplication in $k[G]$ is given by

$$\left(\sum a_\sigma \sigma\right) \left(\sum b_\tau \tau\right) = \left(\sum c_\lambda \lambda\right)$$

where $c_\lambda \in k$ can be given in three different ways:

$$c_\lambda = \sum_{\sigma\tau=\lambda} a_\sigma b_\tau = \sum_{\sigma \in G} a_\sigma b_{\sigma^{-1}\lambda} = \sum_{\tau \in G} a_{\lambda\tau^{-1}} b_\tau$$

Proposition 1.5. $k[G]$ is a k -algebra.

This is straightforward and tedious. So, I didn't prove it. But I did explain what it means and why it is important.

Recall that an *algebra* over k is a ring which contains k in its center. The *center* $Z(R)$ of a (noncommutative) ring R is defined to be the set of elements of R which commute with all the other elements:

$$Z(R) := \{x \in R \mid xy = yx \quad \forall y \in R\}$$

$Z(R)$ is a subring of R .

The center is important for the following reason. Suppose that M is a (left) R -module. Then each element $r \in R$ acts on M by left multiplication λ_r

$$\lambda_r : M \rightarrow M, \quad \lambda_r(x) = rx$$

This is a homomorphism of $Z(R)$ -modules since:

$$\lambda_r(ax) = rax = arx = a\lambda_r(x) \quad \forall a \in Z(R)$$

Thus the action of R on M gives a ring homomorphism:

$$\rho : R \rightarrow \text{End}_{Z(R)}(M)$$

Getting back to $k[G]$, suppose that M is a $k[G]$ -module. Then the action of $k[G]$ on M is k -linear since k is in the center of $k[G]$. So, we get a ring homomorphism

$$\rho : k[G] \rightarrow \text{End}_k(M)$$

This restricts to a group homomorphism

$$\rho|_G : G \rightarrow \text{Aut}_k(M)$$

I pointed out that, in general, any ring homomorphism $\phi : R \rightarrow S$ will induce a group homomorphism $U(R) \rightarrow U(S)$ where $U(R)$ is the group of units of R . And I pointed out earlier that $\text{Aut}_k(M)$ is the group of units of $\text{End}_k(M)$. G is contained in the group of units of $k[G]$. (An interesting related question is: Which finite groups occur as groups of units of rings?)

This discussion shows that a $k[G]$ -module M gives, by restriction, a representation of the group G on the k -vector space M . Conversely, suppose that

$$\rho : G \rightarrow \text{Aut}_k(V)$$

is a group representation. Then we can extend ρ to a ring homomorphism

$$\bar{\rho} : k[G] \rightarrow \text{End}_k(V)$$

by the simple formula

$$\bar{\rho} \left(\sum a_\sigma \sigma \right) = \sum a_\sigma \rho(\sigma)$$

When we say that a representation of a group G is “the same” as a $k[G]$ -module we are talking about this correspondence. The vector space V is also called a G -module. So, it would be more accurate to say that a G -module is the same as a $k[G]$ -module.

Corollary 1.6. (1) *Any group representation $\rho : G \rightarrow \text{Aut}_k(V)$ extends uniquely to a ring homomorphism $\bar{\rho} : k[G] \rightarrow \text{End}_k(V)$ making V into a $k[G]$ -module.*

- (2) For any $k[G]$ -module M , the action of $k[G]$ on M restricts to give a group representation $G \rightarrow \text{Aut}_k(M)$.
- (3) These two operations are inverse to each other in the sense that ρ is the restriction of $\bar{\rho}$ and an action of the ring $k[G]$ is the unique extension of its restriction to G .

There are some conceptual differences between the group representation and the corresponding $k[G]$ -module. For example, the module might not be faithful even if the group representation is:

Definition 1.7. A group representation $\rho : G \rightarrow \text{Aut}_k(V)$ is called faithful if only the trivial element of G acts as the identity on V . I.e., if the kernel of ρ is trivial. An R -module M is called faithful if the annihilator of M is zero. ($\text{ann}(M) = \{r \in R \mid rx = 0 \ \forall x \in M\}$).

These two definitions do not agree. For example, take the representation

$$\rho : \mathbb{Z}/3 \hookrightarrow GL_2(\mathbb{R})$$

which we discussed earlier. This is faithful. But the extension to a ring homomorphism

$$\bar{\rho} : \mathbb{R}[\mathbb{Z}/3] \rightarrow \text{Mat}_2(\mathbb{R})$$

is not a monomorphism since $1 + \sigma + \sigma^2$ is in its kernel.

1.3. Semisimplicity of $k[G]$. The main theorem about $k[G]$ is the following.

Theorem 1.8 (Maschke). *If G is a finite group of order $|G| = n$ and k is a field with $\text{char } k \nmid n$ (or $\text{char } k = 0$) then $k[G]$ is semisimple.*

Instead of saying $\text{char } k$ is either 0 or a prime not dividing n , I will say that $1/n \in k$. By the Wedderburn structure theorem we get the following.

Corollary 1.9. *If $1/|G| \in k$ then*

$$k[G] \cong \text{Mat}_{d_1}(D_1) \times \cdots \times \text{Mat}_{d_b}(D_b)$$

where D_i are finite dimensional division algebras over k .

Example 1.10.

$$\mathbb{R}[\mathbb{Z}/3] \cong \mathbb{R} \times \mathbb{C}$$

In general, if G is abelian, then the numbers d_i must all be 1 and D_i must be finite field extensions of k .

1.3.1. *homomorphisms.* In order to prove Maschke's theorem, we need to talk about homomorphisms of G -modules. We can define these to be the same as homomorphisms of $k[G]$ -modules. Then the following is a proposition. (Or, we can take the following as the definition of a G -module homomorphism, in which case the proposition is that G -module homomorphisms are the same as homomorphisms of $k[G]$ -modules.)

Proposition 1.11. *Suppose that V, W are $k[G]$ -modules. Then a k -linear mapping $\phi : V \rightarrow W$ is a homomorphism of $k[G]$ -modules if and only if it commutes with the action of G . I.e., if*

$$\sigma(\phi(v)) = \phi(\sigma v)$$

for all $\sigma \in G$.

Proof. Any homomorphism of $k[G]$ -modules will commute with the action of $k[G]$ and therefore with the action of $G \subset k[G]$. Conversely, if $\phi : V \rightarrow W$ commutes with the action of G then, for any $\sum a_\sigma \sigma \in k[G]$, we have

$$\phi \left(\sum_{\sigma \in G} a_\sigma \sigma v \right) = \sum_{\sigma \in G} a_\sigma \phi(\sigma v) = \sum_{\sigma \in G} a_\sigma \sigma \phi(v) = \left(\sum_{\sigma \in G} a_\sigma \sigma \right) \phi(v)$$

So, ϕ is a homomorphism of $k[G]$ -modules. \square

We also have the following Proposition/Definition of a G -submodule.

Proposition 1.12. *A subset W of a G -module V over k is a $k[G]$ -submodule (and we call it a G -submodule) if and only if*

- (1) W is a vector subspace of V and
- (2) W is invariant under the action of G . I.e., $\sigma W \subseteq W$ for all $\sigma \in G$.

Proof of Maschke's Theorem. Suppose that V is a finitely generated G -module and W is any G -submodule of V . Then we want to show that W is a direct summand of V . This is one of the characterizations of semisimple modules. This will prove that all f.g. $k[G]$ -modules are semisimple and therefore $k[G]$ is a semisimple ring.

Since W is a submodule of V , it is in particular a vector subspace of V . So, there is a linear projection map $\phi : V \rightarrow W$ so that $\phi|_W = id_W$. If ϕ is a homomorphism of G -modules, then $V = W \oplus \ker \phi$ and W would split from V . So, we would be done. If ϕ is not a G -homomorphism, we can make it into a G -homomorphism by "averaging over the group," i.e., by replacing it with $\psi = \frac{1}{n} \sum \lambda_{\sigma^{-1}} \circ \phi \circ \lambda_\sigma$.

First, I claim that $\psi|_W = id_W$. To see this take any $w \in W$. Then $\sigma w \in W$. So, $\phi(\sigma w) = \sigma w$ and

$$\psi(w) = \frac{1}{n} \sum_{\sigma \in G} \sigma^{-1} \phi(\sigma w) = \frac{1}{n} \sum_{\sigma \in G} \sigma^{-1}(\sigma w) = w$$

Next I claim that ψ is a homomorphism of G -modules. To show this take any $\tau \in G$ and $v \in V$. Then

$$\begin{aligned} \psi(\tau v) &= \frac{1}{n} \sum_{\sigma \in G} \sigma^{-1} \phi(\sigma \tau v) = \frac{1}{n} \sum_{\alpha \beta = \tau} \alpha \phi(\beta v) \\ &= \frac{1}{n} \sum_{\sigma \in G} \tau \sigma^{-1} \phi(\sigma v) = \tau \psi(v) \end{aligned}$$

So, ψ gives a splitting of V as required. □

1.3.2. $\mathbb{R}[\mathbb{Z}/3]$. I gave a longwinded explanation of Example 1.10 using the universal property of the group ring $k[G]$. In these notes, I will just summarize this property in one equation. If R is any k -algebra and $U(R)$ is the group of units of R , then:

$$\text{Hom}_{k\text{-alg}}(k[G], R) \cong \text{Hom}_{grp}(G, U(R))$$

The isomorphism is given by restriction and linear extension.

The isomorphism $\mathbb{R}[\mathbb{Z}/3] \cong \mathbb{R} \times \mathbb{C}$ is given by the mapping:

$$\phi : \mathbb{Z}/3 \rightarrow \mathbb{R} \times \mathbb{C}$$

which sends the generator σ to $(1, \omega)$ where ω is a primitive third root of unity. Since $(1, 0), (1, \omega), (1, \bar{\omega})$ are linearly independent over \mathbb{R} , the linear extension $\bar{\phi}$ of ϕ is an isomorphism of \mathbb{R} -algebras.

1.3.3. *group rings over \mathbb{C}* . We will specialize to the case $k = \mathbb{C}$. In that case, there are no finite dimensional division algebras over \mathbb{C} (Part C, Theorem 3.12). So, we get only matrix algebras:

Corollary 1.13. *If G is any finite group then*

$$\mathbb{C}[G] \cong \text{Mat}_{d_1}(\mathbb{C}) \times \cdots \times \text{Mat}_{d_b}(\mathbb{C})$$

In particular, $n = |G| = \sum d_i^2$.

Example 1.14. *If G is a finite abelian group of order n then $\mathbb{C}[G] \cong \mathbb{C}^n$.*

Example 1.15. *Take $G = S_3$, the symmetric group on 3 letters. Since this group is nonabelian, the numbers d_i cannot all be equal to 1. But*

the only way that 6 can be written as a sum of squares, not all 1, is $6 = 1 + 1 + 4$. Therefore,

$$\mathbb{C}[S_3] \cong \mathbb{C} \times \mathbb{C} \times \text{Mat}_2(\mathbb{C})$$

This can be viewed as a subalgebra of $\text{Mat}_4(\mathbb{C})$ given by

$$\begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$$

Each star (*) represents an independent complex variable. In this description, it is easy to visualize what are the simple factors $\text{Mat}_{d_i}(\mathbb{C})$ given by the Wedderburn structure theorem. But what are the corresponding factors of the group ring $\mathbb{C}[G]$?

1.4. idempotents. Suppose that $R = R_1 \times R_2 \times R_3$ is a product of three subrings. Then the unity of R decomposes as $1 = (1, 1, 1)$. This can be written as a sum of unit vectors:

$$1 = (1, 0, 0) + (0, 1, 0) + (0, 0, 1) = e_1 + e_2 + e_3$$

This is a decomposition of unity (1) as a sum of central, orthogonal idempotents e_i .

Recall that *idempotent* means that $e_i^2 = e_i$ for all i . Also, 0 is not considered to be an idempotent. *Orthogonal* means that $e_i e_j = 0$ if $i \neq j$. *Central* means that $e_i \in Z(R)$.

Theorem 1.16. *A ring R can be written as a product of b subrings R_1, R_2, \dots, R_b iff $1 \in R$ can be written as a sum of b central, orthogonal idempotents and, in that case, $R_i = e_i R$.*

A central idempotent e is called *primitive* if it cannot be written as a sum of two central orthogonal idempotents.

Corollary 1.17. *The number of factors $R_i = e_i R$ is maximal iff each e_i is primitive.*

So, the problem is to write unity $1 \in \mathbb{C}[G]$ as a sum of primitive, central (\Rightarrow orthogonal) idempotents. We will derive a formula for this decomposition using characters.

1.5. **Center of $\mathbb{C}[G]$.** Before I move on to characters, I want to prove one last thing about the group ring $\mathbb{C}[G]$.

Theorem 1.18. *The number of factors b in the decomposition*

$$\mathbb{C}[G] \cong \prod_{i=1}^b \text{Mat}_{d_i}(\mathbb{C})$$

is equal to the number of conjugacy classes of elements of G .

For, example, the group S_3 has three conjugacy classes: the identity $\{1\}$, the transpositions $\{(12), (23), (13)\}$ and the 3-cycles $\{(123), (132)\}$.

In order to prove this we note that b is the dimension of the center of the right hand side. Any central element of $\text{Mat}_{d_i}(\mathbb{C})$ is a scalar multiple of the unit matrix which we are calling e_i (the i th primitive central idempotent). Therefore:

Lemma 1.19. *The center of $\prod_{i=1}^b \text{Mat}_{d_i}(\mathbb{C})$ is the vector subspace spanned by the primitive central idempotents e_1, \dots, e_b . In particular it is b -dimensional.*

So, it suffices to show that the dimension of the center of $\mathbb{C}[G]$ is equal to the number of conjugacy classes of elements of G . (If G is abelian, this is clearly true.)

Definition 1.20. *A class function on G is a function $f : G \rightarrow X$ so that f takes the same value on conjugate elements. I.e.,*

$$f(\tau\sigma\tau^{-1}) = f(\sigma)$$

for all $\sigma, \tau \in G$. Usually, $X = \mathbb{C}$.

For example, any function on an abelian group is a class function.

Lemma 1.21. *For any field k , the center of $k[G]$ is the set of all $\sum_{\sigma \in G} a_\sigma \sigma$ so that a_σ is a class function on G . So, $Z(k[G]) \cong k^c$ where c is the number of conjugacy classes of elements of G .*

Proof. If $\sum_{\sigma \in G} a_\sigma \sigma$ is central then

$$\sum_{\sigma \in G} a_\sigma \sigma = \tau \sum_{\sigma \in G} a_\sigma \sigma \tau^{-1} = \sum_{\sigma \in G} a_\sigma \tau \sigma \tau^{-1}$$

The coefficient of $\tau\sigma\tau^{-1}$ on both sides must agree. So

$$a_{\tau\sigma\tau^{-1}} = a_\sigma$$

I.e., a_σ is a class function. The converse is also clear. □

These two lemmas clearly imply Theorem 1.18 (which can now be stated as: $b = c$ if $k = \mathbb{C}$).

2. CHARACTERS

If $\rho : G \rightarrow GL_d(\mathbb{C})$ is a representation of G over \mathbb{C} then the *character* of ρ is the function

$$\chi_\rho : G \rightarrow \mathbb{C}$$

given by $\chi_\rho(\sigma) = \text{Tr}(\rho(\sigma))$.

The main property of characters is that they determine the representation uniquely up to isomorphism. So, once we find all the characters (by constructing the character table) we will in some sense know all the representations. We will assume that all groups G are finite and all representations are finite dimensional over \mathbb{C} .

2.1. Basic properties. The basic property of trace is that it is invariant under conjugation:

$$\text{Tr}(ABA^{-1}) = \text{Tr}(B)$$

Letting $A = \rho(\sigma)$, $B = \rho(\tau)$ we get

$$\chi_\rho(\sigma\tau\sigma^{-1}) = \text{Tr}(\rho(\sigma\tau\sigma^{-1})) = \text{Tr}(\rho(\sigma)\rho(\tau)\rho(\sigma)^{-1}) = \text{Tr}(\rho(\tau)) = \chi_\rho(\tau)$$

for any representation ρ . So:

Theorem 2.1. *Characters are class functions. (They have the same value on conjugate elements.)*

If $\rho : G \rightarrow \text{Aut}_{\mathbb{C}}(V)$ is a representation of G over \mathbb{C} , then the *character* of ρ , also called the *character of V* , is defined to be the function

$$\chi_\rho = \chi_V : G \rightarrow \mathbb{C}$$

given by

$$\chi_V(\sigma) = \text{Tr}(\rho(\sigma)) = \text{Tr}(\phi \circ \rho(\sigma) \circ \phi^{-1})$$

for any linear isomorphism $\phi : V \xrightarrow{\cong} \mathbb{C}^d$.

There are three basic formulas that I want to explain. In order of difficulty they are:

- (1) The character of a direct sum is the sum of the characters:

$$\chi_{V \oplus W} = \chi_V + \chi_W$$

- (2) The character of a tensor product is the product of the characters:

$$\chi_{V \otimes W} = \chi_V \chi_W$$

- (3) The character of the dual representation is the complex conjugate of the original character:

$$\chi_{V^*} = \overline{\chi_V}$$

2.1.1. *direct sum.* The trace of a direct sum of matrices is the sum of traces:

$$\mathrm{Tr}(A \oplus B) = \mathrm{Tr} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \mathrm{Tr}(A) + \mathrm{Tr}(B)$$

Theorem 2.2. *If V, W are two G -modules then*

$$\chi_{V \oplus W} = \chi_V + \chi_W$$

Proof. If $\rho_V, \rho_W, \rho_{V \oplus W}$ are the corresponding representations then

$$\rho_{V \oplus W}(\sigma) = \rho_V(\sigma) \oplus \rho_W(\sigma)$$

The theorem follows. □

2.1.2. *character formula using dual basis.* Instead of using traces of matrices, I prefer the following equivalent formula for characters using bases and dual bases.

If V is a G -module, we choose a basis $\{v_1, \dots, v_d\}$ for V as a vector space over \mathbb{C} . Then recall that the *dual basis* for $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ consists of the dual vectors $v_1^*, \dots, v_d^* : V \rightarrow \mathbb{C}$ given by

$$v_j^* \left(\sum_{i=1}^d a_i v_i \right) = a_j$$

I.e., v_j^* picks out the coefficient of v_j .

Proposition 2.3.

$$\chi_V(\sigma) = \sum_{i=1}^d v_i^*(\sigma v_i)$$

Proof. The matrix of the linear transformation $\rho(\sigma)$ has (i, j) entry $v_i^*(\sigma v_j)$. Therefore, its trace is $\sum v_i^*(\sigma v_i)$. □

For example, the trace of the identity map is

$$\mathrm{Tr}(id_V) = \sum_{i=1}^d v_i^*(v_i) = d$$

Theorem 2.4. *The value of the character at 1 is the dimension of the representation:*

$$\chi_V(1) = d = \dim_{\mathbb{C}}(V)$$

2.1.3. *tensor product.* If V, W are two G -modules then the *tensor product* $V \otimes W$ is defined to be the tensor product over \mathbb{C} with the following action of G :

$$\sigma(v \otimes w) = \sigma v \otimes \sigma w$$

Theorem 2.5. *The character of $V \otimes W$ is the product of the characters of V and W . I.e.,*

$$\chi_{V \otimes W}(\sigma) = \chi_V(\sigma)\chi_W(\sigma)$$

for all $\sigma \in G$.

Proof. Choose bases $\{v_i\}, \{w_j\}$ for V, W with dual bases $\{v_i^*\}, \{w_j^*\}$. Then the tensor product $V \otimes W$ has basis elements $v_i \otimes w_j$ with dual basis elements $v_i^* \otimes w_j^*$. So, the character is:

$$\begin{aligned} \chi_{V \otimes W}(\sigma) &= \sum_{i,j} (v_i^* \otimes w_j^*) \sigma(v_i \otimes w_j) = \sum_{i,j} (v_i^* \otimes w_j^*) (\sigma v_i \otimes \sigma w_j) \\ &= \sum_{i,j} v_i^*(\sigma v_i) w_j^*(\sigma w_j) = \sum_i v_i^*(\sigma v_i) \sum_j w_j^*(\sigma w_j) = \chi_V(\sigma)\chi_W(\sigma) \end{aligned}$$

□

2.1.4. *dual representation.* The dual space V^* is a right G -module. In order to make it a left G -module we have to invert the elements of the group. I.e., for all $f \in V^*$ we define

$$(\sigma f)(v) := f(\sigma^{-1}v)$$

Lemma 2.6.

$$\chi_{V^*}(\sigma) = \chi_V(\sigma^{-1})$$

Lemma 2.7.

$$\chi_V(\sigma^{-1}) = \overline{\chi_V(\sigma)}$$

Proof. The trace of a matrix A is equal to the sum of its eigenvalues λ_i . If A has finite order: $A^m = I_d$ then its eigenvalues are roots of unity. Therefore, their inverses are equal to their complex conjugates. So,

$$\mathrm{Tr}(A^{-1}) = \sum \lambda_i^{-1} = \sum \bar{\lambda}_i = \overline{\mathrm{Tr}(A)}$$

Since G is finite, the lemma follows. □

Theorem 2.8. *The character of the dual representation V^* is the complex conjugate of the character of V :*

$$\chi_{V^*}(\sigma) = \overline{\chi_V(\sigma)}$$

2.2. Irreducible characters.

Definition 2.9. A representation $\rho : G \rightarrow \text{Aut}_{\mathbb{C}}(V)$ is called irreducible if V is a simple G -module. The character

$$\chi_{\rho} = \chi_V : G \rightarrow \mathbb{C}$$

of an irreducible representation is called an irreducible character.

Theorem 2.10. Every character is a nonnegative integer linear combination of irreducible characters.

Proof. Since $\mathbb{C}[G]$ is semisimple, any G -module V is a direct sum of simple modules $V \cong \bigoplus S_{\alpha}$. So, the character of V is a sum of the corresponding irreducible characters: $\chi_V = \sum \chi_{S_{\alpha}}$. \square

If we collect together multiple copies of the same simple module we get $V = \bigoplus n_i S_i$ and

$$\chi_V = \sum_{i=1}^r n_i \chi_i$$

where χ_i is the character of S_i . This makes sense only if we know that there are only finitely many nonisomorphic simple modules S_i and that the corresponding characters χ_i are distinct functions $G \rightarrow \mathbb{C}$. In fact we will show the following.

Theorem 2.11. (1) There are exactly b (the number of blocks) irreducible representations S_i up to isomorphism.
 (2) The corresponding characters χ_i are linearly independent.

This will immediately imply the following.

Corollary 2.12. The irreducible characters χ_1, \dots, χ_b form a basis for the b -dimensional vector space of all class functions $G \rightarrow \mathbb{C}$.

2.2.1. *regular representation.* This is a particularly elementary representation and character which contains all the simple modules.

Definition 2.13. The free module ${}_{\mathbb{C}[G]}\mathbb{C}[G]$ is called the regular representation of G . The corresponding character is called the regular character: $\chi_{reg} = \chi_{\mathbb{C}[G]} : G \rightarrow \mathbb{C}$.

Theorem 2.14. $\chi_{reg}(\sigma) = \begin{cases} n = |G| & \text{if } \sigma = 1 \\ 0 & \text{if } \sigma \neq 1 \end{cases}$

Proof. I used the basis-dual basis formula for characters. The regular representation $V = \mathbb{C}[G]$ has basis elements $\sigma \in G$ and dual basis elements σ^* given by

$$\sigma^* \left(\sum a_\tau \tau \right) = a_\sigma$$

I.e., $\sigma^*(x)$ is the coefficient of σ in the expansion of x . The regular character is then given by

$$\chi_{reg}(\tau) = \sum_{\sigma \in G} \sigma^*(\tau\sigma)$$

But this is clearly equal to 0 if $\tau \neq 1$ since the coefficient of σ in $\tau\sigma$ is 0. And we already know that $\chi_{reg}(1) = \dim \mathbb{C}[G] = n$. \square

Lemma 2.15. *There are only finitely many isomorphism classes of simple G -modules.*

Proof. First choose a decomposition of the regular representation into simple modules:

$$\mathbb{C}[G] \cong \bigoplus S_\alpha$$

Then I claim that any simple module S is isomorphic to one of the S_α in this decomposition. And this will prove the lemma.

To prove the claim, choose any nonzero element $x_0 \in S$. Then x_0 generates S (the submodule generated by x_0 is either 0 or S). Therefore, we have an epimorphism

$$\phi : \mathbb{C}[G] \twoheadrightarrow S$$

given by $\phi(r) = rx_0$. When we restrict ϕ to each simple component S_α , we get a homomorphism $\phi|_{S_\alpha} : S_\alpha \rightarrow S$ which, by Schur's lemma, must either be zero or an isomorphism. These restrictions cannot all be zero since ϕ is an epimorphism. Therefore, one of them is an isomorphism $S_\alpha \cong S$. This proves the claim. \square

This proof shows more than the lemma states. It proves:

Lemma 2.16. *The regular representation $\mathbb{C}[G]$ contains an isomorphic copy of every simple G -module.*

Therefore, in order to find all the irreducible representations, we need to decompose the regular representation as a sum of simple modules.

2.2.2. *decomposition of the regular representation.* At this point I used the Wedderburn structure theorem again:

$$\mathbb{C}[G] \cong \prod_{i=1}^b \text{Mat}_{d_i}(\mathbb{C}) = \prod R_i$$

where, following Lang, we write $R_i = \text{Mat}_{d_i}(\mathbb{C})$.

Let $S_i = \mathbb{C}^{d_i}$ be the vector space of column vectors. Then R_i acts on the left by matrix multiplication and it is easy to see that S_i is a simple R_i -module since it is generated by any nonzero element.

Lemma 2.17. *If $\phi : R' \rightarrow R$ is an epimorphism of rings and S is any simple R -module, then S becomes a simple R' -module with the action of R' induced by ϕ .*

Proof. If $x_0 \in S$ is any nonzero element then $R'x_0 = Rx_0 = S$. So, any nonzero element of S generates the whole thing as an R' -module. So, it is simple. \square

Since $\mathbb{C}[G] = \prod R_i$, we can make S_i into a G -module with the ring homomorphism:

$$\bar{\rho}_i : \mathbb{C}[G] \xrightarrow{\pi_i} R_i \xrightarrow{\cong} \text{End}_{\mathbb{C}}(S_i)$$

Since $\pi_i : \mathbb{C}[G] \rightarrow R_i$ is an epimorphism, S_i becomes a simple G -module. In other words, the corresponding representation is irreducible:

$$\rho_i : G \rightarrow \text{Aut}_{\mathbb{C}}(S_i)$$

Also, Lang points out that

$$R_j S_i = 0$$

if $i \neq j$. (And $R_i S_i = S_i$.) This is the key point. It shows immediately that the G -modules S_i are not isomorphic. And it will also show that the characters are linearly independent.

In order to show that the characters

$$\chi_i = \chi_{\rho_i} = \chi_{S_i}$$

are linearly independent we will evaluate them on the central idempotents e_i corresponding to the decomposition $\mathbb{C}[G] = \prod R_i$. As we discussed earlier, this product decomposition gives a decomposition of unity:

$$1 = e_1 + \cdots + e_b$$

where e_i is the unity of R_i . (We want to say “ $e_i = 1$ ” but there would be too many 1’s.) We then need to compute $\chi_i(e_j)$. But this is not defined since e_j is not an element of G . We need to extend χ_i to a map on $\mathbb{C}[G]$.

2.2.3. *linear extension of characters.* If $\chi : G \rightarrow \mathbb{C}$ is any character, we define the *linear extension* of χ to $\mathbb{C}[G]$ by the formula

$$\chi\left(\sum a_\sigma \sigma\right) = \sum a_\sigma \chi(\sigma)$$

Since the symbol $\bar{\chi}$ is already taken ($\bar{\chi}$ is the complex conjugate of χ), I decided to use the same symbol χ to denote the linear extension of χ given by the above formula.

The linear extension of χ_ρ is $\chi_{\bar{\rho}}$ which is the trace of the linear extension of ρ . To see this let $x = \sum a_\sigma \sigma$. Then

$$\sum a_\sigma \chi_\rho(\sigma) = \sum a_\sigma \text{Tr}(\rho(\sigma)) = \text{Tr}\left(\sum a_\sigma \rho(\sigma)\right) = \text{Tr}(\bar{\rho}(x))$$

Lemma 2.18.

$$\chi_i(e_j) = \begin{cases} 0 & \text{if } i \neq j \\ d_i & \text{if } i = j \end{cases}$$

Proof. If $i \neq j$ we have

$$\chi_i(e_j) = \text{Tr}(\rho_i(e_j)) = 0$$

since $\rho_i(e_j)$ is the zero matrix (giving the action of $e_j \in R_j$ on W_i).

If $i = j$ then

$$\chi_i(e_i) = \dim S_i = d_i$$

since e_i is unity in R_i . □

This proves the second part of Theorem 2.11: If $\sum a_i \chi_i = 0$ then

$$\sum a_i \chi_i(e_j) = a_j d_j = 0$$

which forces $a_j = 0$ for all j .

Theorem 2.19. *The regular representation decomposes as:*

$$\mathbb{C}[G] \cong \sum_{i=1}^b d_i S_i$$

Proof. The i th block of the Wedderburn decomposition is a $d_i \times d_i$ matrix which, as a left module, decomposes into d_i column vectors, i.e., into a direct sum of d_i copies of the simple module d_i . □

2.2.4. *example.* Take $G = S_3$. Then we already saw that

$$\mathbb{C}[S_3] \cong \mathbb{C} \times \mathbb{C} \times \text{Mat}_2(\mathbb{C})$$

So, there are three simple modules S_1, S_2, S_3

$S_1 = \mathbb{C}$ is the trivial representation.

S_2 is the sign representation $\rho_2(\sigma) = \text{sgn}(\sigma) = \pm 1$

S_3 is a simple 2-dimensional module.

Since characters are class functions, their value is the same on conjugate elements. So, we only need their values on representatives $1, (12), (123)$. The characters χ_1, χ_2 are easy to compute. The last irreducible character is determined by the equation

$$\chi_{reg} = \chi_1 + \chi_2 + 2\chi_3$$

So, here is the character table of S_3 :

	1	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1
<hr/>			
χ_{reg}	6	0	0

All characters of S_3 are nonnegative integer linear combinations of χ_1, χ_2, χ_3 .

2.3. formula for idempotents. Lang gives a formula for the idempotents $e_i \in \mathbb{C}[G]$ in terms of the corresponding irreducible character χ_i . The key point is that the linear extension

$$\bar{\rho}_i : \mathbb{C}[G] \rightarrow \text{End}_{\mathbb{C}}(S_i) = \text{Mat}_{d_i}(\mathbb{C})$$

of ρ_i sends e_i to the identity matrix. Therefore,

$$\bar{\rho}_i(e_i\sigma) = \bar{\rho}_i(e_i)\rho_i(\sigma) = \rho_i(\sigma)$$

Also, $\bar{\rho}_i(e_j\sigma)$ is the zero matrix if $i \neq j$. Therefore,

$$\chi_i(e_j\sigma) = \begin{cases} \rho_i(\sigma) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Now use the regular character. If $e_i = \sum a_{\tau}\tau$ then

$$a_{\tau} = \frac{1}{n}\chi_{\text{reg}}(e_i\tau^{-1}) = \frac{1}{n}\sum_j d_j\chi_j(e_i\tau^{-1}) = \frac{d_i}{n}\chi_i(\tau^{-1})$$

Theorem 2.20.

$$e_i = \frac{d_i}{n}\sum_{\tau \in G}\chi_i(\tau^{-1})\tau$$

This formula has an important consequence.

Corollary 2.21. $d_i|n$ (Each d_i divides $n = |G|$.)

Proof. First recall that e_i is an idempotent. So,

$$e_i = e_i^2 = \frac{d_i}{n}\sum_{\tau \in G}\chi_i(\tau^{-1})\tau e_i$$

Now multiply by n/d_i to get:

$$(2.1) \quad \frac{n}{d_i}e_i = \sum_{\tau \in G}\chi_i(\tau^{-1})\tau e_i$$

I mentioned earlier that $\chi_i(\tau^{-1}) = \sum \lambda_j$ is a sum of m th roots of unity where $m = o(\tau^{-1}) = o(\tau)$. But this number divides $n = |G|$. So, each λ_j is a power of $\zeta = e^{2\pi i/n}$

Let $M_i \subset \mathbb{C}[G]$ be the additive subgroup generated by all elements of the form $\zeta^j\tau e_i$ (for all j and fixed i). This is a finitely generated torsion free (and thus free) \mathbb{Z} -module and equation (2.1) shows that M_i is invariant under multiplication by the rational number n/d_i . Therefore, n/d_i is integral. Since \mathbb{Z} is integrally closed in \mathbb{Q} this implies that $n/d_i \in \mathbb{Z}$. \square

2.4. character tables. I decided to construct some character tables (as I did for $G = S_3$) and explain properties of characters using the examples. The *character table* is defined to be the $b \times b$ matrix with entries $\chi_i(c_j)$ where c_j is the j th conjugacy class. The characters are usually arranged in order of degree d_i with χ_1 being the trivial character. The conjugacy classes are arranged arbitrarily with $c_1 = \{1\}$. So, the character table looks like this:

	1	c_2	c_3	\cdots	c_b
χ_1	1	1	1	\cdots	1
χ_2	d_2				
χ_3	d_3				
\cdots	\cdots			$\chi_i(c_j)$	
χ_b	d_b				

2.4.1. one-dimensional characters. The case $d = 1$ is very special. First of all, any one-dimensional representation of G is irreducible. So, it is one of the ρ_i . Here are all the things I pointed out:

Proposition 2.22. *Suppose that $d_i = 1$. Then*

- (1) $\chi_i = \rho_i$: *The character is the representation.*
- (2) $\chi_i(\sigma)$ *is an m th root of unity where $m = o(\sigma)$.*
- (3) $\chi_i(\sigma\tau) = \chi_i(\sigma)\chi_i(\tau)$.

Proof. This hardly need proof. When $d_i = 1$, the representation is:

$$\rho_i : G \rightarrow \text{Aut}_{\mathbb{C}}(S_i) = GL_1(\mathbb{C}) = \mathbb{C}^\times$$

The trace of a 1×1 matrix is equal to the matrix itself. So, $\chi_i(\sigma) = \rho_i(\sigma)$. Since ρ_i is a homomorphism, so is χ_i . This means χ_i is multiplicative. Also, $\sigma^m = 1$ implies that $\chi_i(\sigma)^m = 1$. □

2.4.2. example: $\mathbb{Z}/3$. Since $\mathbb{Z}/3 = \{1, \sigma, \sigma^2\}$ is an abelian group we have $b = c = n = 3$. Every element is its own conjugacy class. Also, all blocks have size $d_i = 1$. This gives the following partial character table.

	1	σ	σ^2
χ_1	1	1	1
χ_2	1		
χ_3	1		

From our discussion of one-dimensional characters we know that each $\chi_i(\sigma)$ is a third root of unity:

$$\chi_i(\sigma) = 1, \omega, \omega^2$$

$\chi_i(\sigma^2) = \chi_i(\sigma)^2 = 1, \omega^2, \omega$ respectively. So, the complete character table is:

	1	σ	σ^2
χ_1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω

2.4.3. *example: $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.* Let's call the elements of the group $1, \sigma, \tau, \sigma\tau$. Since $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ is abelian, all characters are again one dimensional and the values must be square roots of 1, i.e., they must be ± 1 . So, we got the following.

	1	σ	τ	$\sigma\tau$
χ_1	1	1	1	1
χ_2	1	-1	1	-1
χ_3	1	-1	-1	1
χ_4	1	1	-1	-1

Each row is clearly a one-dimensional representation. There are no others because we know that there are exactly $b = 4$ such representations. So, this is the complete character table.

2.4.4. *example: D_4 .* This is the *dihedral group* of order 8 with presentation:

$$D_4 = \langle \sigma, \tau \mid \sigma^4, \tau^2, \sigma\tau\sigma\tau \rangle$$

(Replace 4 by any n to get the dihedral group of order $2n$.) To find the numbers d_i we have to write $n = 8$ as a sum of squares which are not all 1 (because D_4 is nonabelian) and so that there is at least one 1 (since $d_1 = 1$). The solution is:

$$8 = 1 + 1 + 1 + 1 + 4$$

Therefore, $b = c = 5$.

The elements of the group are:

$$D_4 = \{e, \sigma, \sigma^2, \sigma^3, \tau, \sigma\tau, \sigma^2\tau, \sigma^3\tau\}$$

Among these, σ, σ^3 are conjugate since $\tau\sigma\tau^{-1} = \sigma^3$, $\tau, \sigma^2\tau = \sigma\tau\sigma^{-1}$ are conjugate and $\sigma\tau, \sigma^3\tau = \sigma(\sigma\tau)\sigma^{-1}$ are conjugate. There are no other conjugacy relations since we got it down to 5 classes.

Among the 5 characters, the first 4 are 1-dimensional. And we can find them very quickly as follows. The center of D_4 is the set of elements which are alone in their conjugacy class. So,

$$Z(D_4) = \{1, \sigma^2\}$$

This is a normal subgroup of D_4 with quotient isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.

We already have four irreducible representations ρ_1, \dots, ρ_4 of $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. We can compose with the projection to get four irreducible representations of D_4

$$D_4 \twoheadrightarrow D_4/Z \xrightarrow{\rho_i = \chi_i} \mathbb{C}^\times$$

This gives the first four lines in the character table:

	1	σ^2	σ	τ	$\sigma\tau$
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	-1	-1	1
χ_4	1	1	1	-1	-1
χ_5	2	-2	0	0	0

To get the last line we use the equation:

$$\chi_{reg} = \sum d_i \chi_i = \chi_1 + \chi_2 + \chi_3 + \chi_4 + 2\chi_5$$

2.4.5. *kernel of a representation.* Looking at the character table, we can determine which elements of the group lie in the kernel of each representation.

Lemma 2.23. $\sigma \in \ker \rho \iff \chi_\rho(\sigma) = d = \chi_\rho(1)$.

Proof. In a d -dimensional representation, $\chi(\sigma) = \lambda_1 + \dots + \lambda_d$ is a sum of d roots of unity. This sum is equal to d if and only if every $\lambda_i = 1$ which is equivalent to saying that $\rho(\sigma)$ is the identity matrix (since $\rho(\sigma)$ has finite order). \square

Using the same argument it follows that:

Proposition 2.24. $|\chi_\rho(\sigma)| = d$ if and only if $\rho(\sigma) = \lambda I_d$ is a scalar multiple of the identity matrix. Furthermore, $\lambda = \chi_\rho(\sigma)/d$.

For example, in the last irreducible representation of D_4 we have

$$|\chi_5(\sigma^2)| = 2 = d_5$$

Therefore, $\rho_5(\sigma^2) = -I_2$.

2.4.6. *finding all normal subgroups.* Finally, I claimed that the character table determines all normal subgroups of the group G . This is based on the trick that we used to construct the character table of D_4 .

Suppose that N is a normal subgroup of G and ρ_i , $i = 1, \dots, r$ are the irreducible representations of G/N .

Lemma 2.25.

$$N = \bigcap \ker(\rho_i \circ \pi)$$

where $\pi : G \rightarrow G/N$ is the quotient map.

Proof. Let $K = \bigcap \ker(\rho_i \circ \pi)$. Then clearly, $N \subseteq K$. So, suppose that K is bigger than N . Then the representations ρ_i would all factor through the quotient G/K :

$$\rho_i : G/N \xrightarrow{\phi} G/K \xrightarrow{\psi_i} \text{Aut}_{\mathbb{C}}(S_i)$$

This is not possible because the sum of the squares of the dimensions of these representations add up to the order of G/N :

$$|G/K| < |G/N| = \sum d_i^2$$

So, the ψ_i are distinct irreducible representations of G/K whose dimensions squared add up to more than the order of the group. This contradiction proves the lemma. \square

Combining Lemmas 2.25 and 2.23, we get the following.

Theorem 2.26. *The normal subgroups of a finite group G can be determined from its character table as follows.*

- (1) *The kernel of ρ_i is the union of all conjugacy classes c_j for which $\chi_i(c_j) = d_i = \chi_i(1)$.*
- (2) *A collection of conjugacy classes forms a normal subgroup if and only if it is an intersection of kernels of irreducible representations ρ_i .*

2.5. orthogonality relations. The character table satisfies two orthogonality relations:

- (1) row orthogonality
- (2) column orthogonality

First, I will do row orthogonality. The rows are the characters χ_i . We want to show that they are “orthogonal” in some sense.

2.5.1. *main theorem and consequences.*

Definition 2.27. If $f, g : G \rightarrow \mathbb{C}$ are class functions then we define $\langle f, g \rangle \in \mathbb{C}$ by

$$\langle f, g \rangle = \langle g, f \rangle = \frac{1}{n} \sum_{\sigma \in G} f(\sigma)g(\sigma^{-1})$$

The main theorem is the following.

Theorem 2.28. If V, W are G -modules then

$$\langle \chi_V, \chi_W \rangle = \dim_{\mathbb{C}} \text{Hom}_G(V, W)$$

Before I prove this let me explain the consequences.

Corollary 2.29. The rows of the character table are orthonormal in the sense that:

$$\langle \chi_i, \chi_j \rangle = \delta_{ij}$$

Proof. It follows from Schur’s lemma that

$$\langle \chi_i, \chi_j \rangle = \dim_{\mathbb{C}} \text{Hom}_G(S_i, S_j) = \delta_{ij}$$

since $\text{Hom}_G(S_i, S_j) = 0$ for $i \neq j$ and $\text{Hom}_{\mathbb{C}}(S_i, S_i) = \mathbb{C}$. □

Since only conjugacy classes appear in the character table we have:

$$\langle \chi_i, \chi_j \rangle = \sum_{k=1}^b |c_k| \chi_i(c_k) \overline{\chi_j(c_k)}$$

For example, for $G = S_3$ we have the character table:

$ c_j $	1	3	2
	1	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

$$\langle \chi_1, \chi_2 \rangle = \frac{(1)(1) + 3(1)(-1) + 2(1)(1)}{6} = \frac{1 - 3 + 2}{6} = 0$$

This formula also tells us that a representation is determined by its character in the following way.

Corollary 2.30. *Suppose that the semisimple decomposition of the G -module V is $V = \sum n_i S_i$. Then*

$$n_i = \langle \chi_V, \chi_i \rangle$$

Proof. Since $\chi_{V \oplus W} = \chi_V + \chi_W$, we have: $\chi_V = \sum n_j \chi_j$. So,

$$\langle \chi_V, \chi_i \rangle = \left\langle \sum n_j \chi_j, \chi_i \right\rangle = n_i$$

□

2.5.2. *proof of the main theorem.* The theorem will follow from three lemmas. The first lemma calculates the dimension of the fixed point set of V .

Definition 2.31. *If V is a G -module then the fixed point set of the action of G is given by*

$$V^G := \{v \in V \mid \sigma v = v \ \forall \sigma \in G\}$$

Lemma 2.32. *The dimension of the fixed point set is equal to the average value of the corresponding character:*

$$\dim_{\mathbb{C}} V^G = \frac{1}{n} \sum_{\sigma \in G} \chi_V(\sigma)$$

Proof. The projection map

$$\pi : V \rightarrow V^G$$

is given by

$$\pi(v) = \frac{1}{n} \sum \sigma v$$

It is clear that

- (1) $\pi(v) \in V^G$ since multiplication by any $\tau \in G$ will just permute the summands.
- (2) $\pi(v) = v$ if $v \in V^G$ because, in that case, each $\sigma v = v$ and there are n terms.

Therefore, π is a projection map, i.e., a linear retraction onto V^G . Looking at the formula we see that π is multiplication by the idempotent $e_1 = \frac{1}{n} \sum_{\sigma \in G} \sigma$. (This is the idempotent corresponding to the trivial representation.) So:

$$\dim V^G = \text{Tr}(\pi) = \chi_V(e_1) = \chi_V \left(\frac{1}{n} \sum_{\sigma \in G} \sigma \right) = \frac{1}{n} \sum_{\sigma \in G} \chi_V(\sigma)$$

Explanations:

(1) $\dim V^G = \text{Tr}(\pi)$ because $V \cong V^G \oplus W$ ($W = \ker \pi$). So, the matrix of π is:

$$\pi = \begin{pmatrix} 1_{V^G} & 0 \\ 0 & 0_W \end{pmatrix}$$

making $\text{Tr}(\pi) = \text{Tr}(1_{V^G}) = \dim_{\mathbb{C}} V^G$.

(2) $\text{Tr}(\pi) = \chi_V(e_1)$ by definition of the character:

$$\chi_V(e_1) := \text{Tr}(e_1 \cdot : V \rightarrow V)$$

This is the trace of the mapping $V \rightarrow V$ given by multiplication by e_1 . But we are calling that mapping π . \square

Lemma 2.33. *If V, W are representations of G then*

$$\text{Hom}_G(V, W) = \text{Hom}_{\mathbb{C}}(V, W)^G$$

where G acts on $\text{Hom}_{\mathbb{C}}(V, W)$ by conjugation, i.e., $\sigma f = \sigma \circ f \circ \sigma^{-1}$ which means that

$$(\sigma f)(v) = \sigma f(\sigma^{-1}v)$$

Proof. This is trivial. Given any linear map $f : V \rightarrow W$, f is a G -homomorphism iff

$$\sigma \circ f = f \circ \sigma \iff \sigma \circ f \circ \sigma^{-1} = f \iff \sigma f = f$$

iff $f \in \text{Hom}_{\mathbb{C}}(V, W)^G$. \square

Lemma 2.34. *$\text{Hom}_{\mathbb{C}}(V, W) \cong V^* \otimes W$ as G -modules.*

Proof. Let $\phi : V^* \otimes W \rightarrow \text{Hom}_{\mathbb{C}}(V, W)$ be given by

$$\phi(f \otimes w)(v) = f(v)w$$

To check that this is a G -homomorphism we need to show that $\phi\sigma = \sigma\phi$ for any $\sigma \in G$. So, we compute both sides:

$$\phi\sigma(f \otimes w) = \phi(\sigma f \otimes \sigma w) = \phi(f \circ \sigma^{-1} \otimes \sigma w)$$

which sends $v \in V$ to

$$\phi(f \circ \sigma^{-1} \otimes \sigma w)(v) = f(\sigma^{-1}v)\sigma w$$

On the other side we have:

$$\sigma\phi(f \otimes w) = \sigma \circ \phi(f \otimes w) \circ \sigma^{-1}$$

which also sends $v \in V$ to

$$\sigma \circ \phi(f \otimes w) \circ \sigma^{-1}v = \sigma(f(\sigma^{-1}v)w) = f(\sigma^{-1}v)\sigma w$$

This shows that ϕ commutes with the action of G . The fact that ϕ is an isomorphism is well-known: If v_i, v_i^* form a basis-dual basis pair for V and w_j form a basis for W then $v_j^* \otimes w_i$ form a basis for $V^* \otimes W$ and

$$\phi(v_j^* \otimes w_i) : v = \sum a_j v_j \mapsto v_j^*(v)w_i = a_j w_i$$

is the mapping whose matrix has ij -entry equal to 1 and all other entries 0. So, these homomorphisms form a basis for $\text{Hom}_{\mathbb{C}}(V, W)$ and ϕ is an isomorphism. \square

Proof of main theorem 2.28. Using the three lemmas we get:

$$\begin{aligned} \dim_{\mathbb{C}} \text{Hom}_G(V, W) &=_{2.33} \dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}}(V, W)^G \\ &=_{2.34} \dim_{\mathbb{C}}(V^* \otimes W)^G \\ &=_{2.32} \frac{1}{n} \sum_{\sigma \in G} \chi_{V^* \otimes W}(\sigma) \\ &= \frac{1}{n} \sum_{\sigma} \chi_{V^*}(\sigma) \chi_W(\sigma) \\ &= \frac{1}{n} \sum_{\sigma} \chi_V(\sigma^{-1}) \chi_W(\sigma) = \langle \chi_V, \chi_W \rangle \end{aligned}$$

\square

2.5.3. *character table of S_4 .* Using these formulas we can calculate the character table for S_4 . First note that there are five conjugacy classes represented by

$$1, (12), (123), (12)(34), (1234)$$

The elements of cycle form $(12)(34)$ form (with 1) a normal subgroup

$$K = \{1, (12)(34), (13)(24), (14)(23)\} \triangleleft S_4$$

called the *Klein 4-group*. The quotient S_4/K is isomorphic to the symmetric group on 3 letters. Imitating the case of D_4 , this allows us

to construct the following portion of the character table for S_4 :

$ c_j $	1	6	8	3	6
	1	(12)	(123)	(12)(34)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	-1	2	0
χ_4	3				
χ_5	3				

Explanations:

- (1) Since $(12)(34) \in K$, the value of the first three characters on this conjugacy class is d_i , the same as in the first column.
- (2) Since $(1234)K = (12)K$, these two columns have the same values of χ_1, χ_2, χ_3 .
- (3) Finally, the two unknown characters χ_4, χ_5 must be 3-dimensional since

$$24 = \sum d_i^2 = 1 + 1 + 4 + d_4^2 + d_5^2$$

has only one solution: $d_4 = d_5 = 3$.

To figure out the unknown characters we need another representation. The *permutation representation* P is the 4-dimensional representation of S_4 in which the elements of S_4 act by permuting the unit coordinate vectors. For example

$$\rho_P(12) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note that the trace of $\rho_P(\sigma)$ is equal to the number of letters left fixed by σ . So, χ_P takes values 4, 2, 1, 0, 0 as shown:

$ c_j $	1	6	8	3	6
	1	(12)	(123)	(12)(34)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	-1	2	0
χ_P	4	2	1	0	0
$\chi_V = \chi_P - \chi_1$	3	1	0	-1	-1

The representation P contains one copy of the trivial representation and no copies of the other two:

$$\begin{aligned}\langle \chi_P, \chi_1 \rangle &= \frac{1}{24}(4 + 6(2) + 8(1)) = 1 \\ \langle \chi_P, \chi_2 \rangle &= \frac{1}{24}(4 + 6(-1)(2) + 8(1)(1)) = 0 \\ \langle \chi_P, \chi_3 \rangle &= \frac{1}{24}((2)(4) + 8(-1)(1)) = 0\end{aligned}$$

So, $P \cong S_1 \oplus V$ where V is a 3-dimensional module which does not contain S_1, S_2 or S_3 . So, $V = nS_4 \oplus mS_5$. But S_4, S_5 are both 3-dimensional. So, $V = S_4$ (or S_5).

Using the fact that

$$\chi_1 + \chi_2 + 2\chi_3 + 3\chi_4 + 3\chi_5 = \chi_{reg}$$

we can now complete the character table of S_4 :

$ c_j $	1	6	8	3	6
	1	(12)	(123)	(12)(34)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	-1	2	0
χ_4	3	1	0	-1	-1
χ_5	3	-1	0	-1	1

From the character table of S_4 we can find all normal subgroups. First, the kernels of the 5 irreducible representations are:

- (1) $\ker \rho_1 = S_4$.
- (2) $\ker \rho_2 = A_4$ containing the conjugacy classes of 1, (123), (12)(34).
- (3) $\ker \rho_3 = K$ containing 1, (12)(34) and conjugates.
- (4) $\ker \rho_4 = 1$. I.e., ρ_4 is a faithful representation.
- (5) $\ker \rho_5 = 1$. So, ρ_5 is also faithful.

Since these subgroups contain each other:

$$1 < K < A_4 < S_4$$

intersecting them will not give any other subgroups. So, these are the only normal subgroups of S_4 .

2.5.4. *column orthogonality.* The columns of the character table also satisfy an orthogonality condition. To see it we first have to write the row orthogonality condition

$$\langle \chi_i, \chi_j \rangle = \sum_{k=1}^b \frac{|c_k|}{n} \chi_i(c_k) \overline{\chi_j(c_k)} = \delta_{ij}$$

and write it in matrix form:

$$T \begin{pmatrix} \frac{|c_1|}{n} & & 0 \\ & \ddots & \\ 0 & & \frac{|c_b|}{n} \end{pmatrix} \overline{T}^t = I_b$$

where T is the character table $T = (\chi_i(c_j))$. This equation shows that the character table T is an invertible matrix with inverse

$$T^{-1} = D \overline{T}^t$$

where D is the diagonal matrix with diagonal entries $\frac{|c_i|}{n}$. Multiplying both sides of this equation on the right by T and on the left with D^{-1} and we get:

$$\overline{T}^t T = D^{-1} = \begin{pmatrix} \frac{n}{|c_1|} & & 0 \\ & \ddots & \\ 0 & & \frac{n}{|c_b|} \end{pmatrix}$$

Looking at the entries of these matrices we get the column orthogonality relation:

Theorem 2.35. *If $\sigma, \tau \in G$ then*

$$\sum_{i=1}^b \overline{\chi_i(\sigma)} \chi_i(\tau) = \begin{cases} \frac{n}{|c|} & \text{if } \sigma, \tau \text{ are conjugate} \\ 0 & \text{if not} \end{cases}$$

Here $|c|$ is the number of conjugates of σ in G . (So, $n/|c|$ is the order of the centralizer $C(\sigma) = \{\tau \in G \mid \sigma\tau = \tau\sigma\}$ of σ .)

Corollary 2.36. *The character table $T = (\chi_i(c_j))$ determines the size of each conjugacy class c_j .*

Proof. Taking $\sigma = \tau$ in the above theorem we get

$$|C(\sigma)| = \sum_i \|\chi_i(\sigma)\|^2$$

The size of the conjugacy class c of σ is the index of its centralizer: $|c| = |G : C(\sigma)| = n/|C(\sigma)|$. \square

As an example, look at the character table for S_3 :

	1	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Column orthogonality means that the usual Hermitian dot product of the columns is zero. For example, the dot product of the first and third column is

$$(1)(1) + (1)(1) + (2)(-1) = 0$$

Also the dot product of the j th vector with itself (its length squared) is equal to $n/|c_j|$. For example, the length squared of the third column vector is

$$1 + 1 + 1 = 3$$

Making the number of conjugates of (123) equal to $6/3 = 2$.

3. INDUCTION

If H is a subgroup of G then any representation of G will restrict to a representation of H by composition:

$$H \hookrightarrow G \xrightarrow{\rho} \text{Aut}_{\mathbb{C}}(V)$$

Induction is a more complicated process which goes the other way: It starts with a representation of H and produces a representation of G . Following Lang, I will construct the same object in several different ways starting with an elementary equation for the induced character.

3.1. induced characters.

Definition 3.1. *Suppose that $H \leq G$ (H is a subgroup of G) and $\chi : H \rightarrow \mathbb{C}$ is a character (or any class function). Then the induced character*

$$\text{Ind}_H^G \chi : G \rightarrow \mathbb{C}$$

is the class function on G defined by

$$\text{Ind}_H^G \chi(\sigma) = \frac{1}{|H|} \sum_{\tau \in G} \chi(\tau \sigma \tau^{-1})$$

where $\chi(\sigma) = 0$ if $\sigma \notin H$.

The main theorem about the induced character is the following.

Theorem 3.2. *If V is any representation of H then there exists a representation W of G so that*

$$\chi_W = \text{Ind}_H^G \chi_V$$

Furthermore, W is unique up to isomorphism.

The representation W is written $W = \text{Ind}_H^G V$ and is called the *induced representation*. We will study that tomorrow.

Before proving this theorem let me give two examples.

3.1.1. *example 1.* Here is a trivial observation.

Proposition 3.3. *If G is abelian then*

$$\text{Ind}_H^G \chi(\sigma) = |G : H| \chi(\sigma)$$

Now suppose that $G = \mathbb{Z}/4 = \{1, \sigma, \sigma^2, \sigma^3\}$ and $H = \{1, \tau\}$ with $\tau = \sigma^2$. Then the character table of $H \cong \mathbb{Z}/2$ is

$H = \mathbb{Z}/2$	1	τ
χ_+	1	1
χ_-	1	-1

I want to calculate $\text{Ind}_{\mathbb{Z}/2}^{\mathbb{Z}/4} \chi_-$. By the proposition, the value of this induced character on $1, \sigma, \sigma^2, \sigma^3$ is the index $|G : H| = 2$ times $1, 0, -1, 0$ respectively. This gives $2, 0, -2, 0$ as indicated below the character table for $G = \mathbb{Z}/4$:

$G = \mathbb{Z}/4$	1	σ	σ^2	σ^3
χ_1	1	1	1	1
χ_2	1	-1	1	-1
χ_3	1	i	-1	$-i$
χ_4	1	$-i$	-1	i
$\text{Ind}_{\mathbb{Z}/2}^{\mathbb{Z}/4} \chi_-$	2	0	-2	0

By examination we see that

$$\text{Ind}_{\mathbb{Z}/2}^{\mathbb{Z}/4} \chi_- = \chi_3 + \chi_4$$

3.1.2. *example 2.* In the nonabelian case we have the following formula which is analogous to the one in the abelian case.

Proposition 3.4.

$$\text{Ind}_H^G \chi(\sigma) = |G : H|(\text{average value of } \chi(\tau\sigma\tau^{-1}))$$

Now let $G = S_3$ and $H = \{1, (12)\} \cong \mathbb{Z}/2$. Using the same notation as in the previous example, let χ_- be the one dimensional character on H given by $\chi_-(1) = 1, \chi_-(12) = -1$. We want to compute the induced character $\text{Ind}_H^G \chi_-$.

$$\text{Ind}_H^G \chi_-(1) = |G : H| \chi_-(1) = (3)(1) = 3$$

Since (12) has three conjugates only one of which lies in H , the average value of χ_1 on these conjugates is

$$\frac{1}{3}(-1 + 0 + 0) = -\frac{1}{3}$$

So,

$$\text{Ind}_H^G \chi_-(12) = |G : H| \left(-\frac{1}{3} \right) = \frac{-3}{3} = -1$$

Since neither of the conjugates of (123) lie in H we have:

$$\text{Ind}_H^G \chi_-(123) = 0$$

So, $\text{Ind}_H^G \chi_-$ takes the values 3, -1, 0 on the conjugacy classes of $G = S_3$. Put it below the character table of S_3 :

$G = S_3$	1	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1
$\text{Ind}_H^G \chi_-$	3	-1	0

We can see that

$$\text{Ind}_H^G \chi_- = \chi_2 + \chi_3$$

3.1.3. *Frobenius reciprocity for characters.* First I need some fancy notation for a very simple concept. If $f : G \rightarrow \mathbb{C}$ is any class function then the *restriction* of f to H , denoted $\text{Res}_H^G f$, is the composition of f with the inclusion map $j : H \hookrightarrow G$:

$$\text{Res}_H^G f = f \circ j : H \rightarrow \mathbb{C}$$

Theorem 3.5 (Frobenius reciprocity). *Suppose that g, h are class functions on G, H respectively. Then*

$$\langle \text{Ind}_H^G h, g \rangle_G = \langle h, \text{Res}_H^G g \rangle_H$$

Suppose for a moment that this is true. Then, letting $h = \chi_V$ and taking g to be the irreducible character $g = \chi_i$, we get:

$$\langle \text{Ind}_H^G \chi_V, \chi_i \rangle_G = \langle \chi_V, \text{Res}_H^G \chi_i \rangle_H = n_i$$

Since $\text{Res}_H^G \chi_i$ is the character of the G -module S_i considered as an H -module, the number n_i is a nonnegative integer, namely:

$$n_i = \dim_{\mathbb{C}} \text{Hom}_H(V, S_i)$$

This implies that

$$\text{Ind}_H^G \chi_V = \chi_W$$

where W is the G -module $W = \bigoplus n_i S_i$. In other words, the induced character is an *effective character* (the character of some representation).

Corollary 3.6. *If $h : H \rightarrow \mathbb{C}$ is an effective character then so is $\text{Ind}_H^G h : G \rightarrow \mathbb{C}$.*

This is a rewording of the main theorem (Theorem 3.2).

Proof of Frobenius reciprocity for characters. Since

$$\text{Ind}_H^G h(\sigma) = \frac{1}{|H|} \sum_{\tau \in G} h(\tau\sigma\tau^{-1})$$

the left hand side of our equation is

$$LHS = \frac{1}{|G|} \sum_{\sigma \in G} \frac{1}{|H|} \sum_{\tau \in G} h(\tau\sigma\tau^{-1})g(\sigma^{-1})$$

Since g is a class function, $g(\sigma^{-1}) = g(\tau\sigma^{-1}\tau^{-1})$. Letting $\alpha = \tau\sigma\tau^{-1}$ we get a sum of terms of the form

$$h(\alpha)g(\alpha^{-1})$$

How many times does each such term occur?

Claim: The number of ways that α can be written as $\alpha = \tau\sigma\tau^{-1}$ is exactly $n = |G|$.

The proof of this claim is simple. For each $\tau \in G$ there is exactly one σ which works, namely, $\sigma = \tau^{-1}\alpha\tau$.

This implies that

$$LHS = \frac{1}{|H|} \sum_{\alpha \in G} h(\alpha)g(\alpha^{-1})$$

Since h is a class function on H , $h(\alpha) = 0$ if $\alpha \notin H$. Therefore, the sum can be restricted to $\alpha \in H$ and this expression is equal to the RHS of the Frobenius reciprocity equation. \square

3.1.4. *examples of Frobenius reciprocity.* Let's take the two example of induced characters that we did earlier and look at what Frobenius reciprocity says about them.

In the case $G = \mathbb{Z}/4$, $H = \mathbb{Z}/2$, the restrictions of the four irreducible characters of $G = \mathbb{Z}/4$ to H (given by the first and third columns) are:

$$\text{Res}_H^G \chi_1 = \chi_+$$

$$\text{Res}_H^G \chi_2 = \chi_+$$

$$\text{Res}_H^G \chi_3 = \chi_-$$

$$\text{Res}_H^G \chi_4 = \chi_-$$

Frobenius reciprocity says that the number of times that χ_- appears in the decomposition of $\text{Res}_H^G \chi_i$ is equal to the number of times that χ_i appears in the decomposition of $\text{Ind}_H^G \chi_-$. So,

$$\text{Ind}_H^G \chi_- = \chi_3 + \chi_4$$

In the case $G = S_3, H = \{1, (12)\}$, the restrictions of the three irreducible characters of $G = S_3$ to H , as given by the first two columns, are:

$$\begin{aligned} \text{Res}_H^G \chi_1 &= \chi_+ \\ \text{Res}_H^G \chi_2 &= \chi_- \\ \text{Res}_H^G \chi_3 &= (2, 0) = \chi_+ + \chi_- \end{aligned}$$

Since χ_- appears once in the restrictions of χ_2, χ_3 we have

$$\text{Ind}_H^G \chi_- = \chi_2 + \chi_3$$

3.1.5. *induction-restriction tables.* The results of the calculations in these two examples are summarized in the following tables which are called *induction-restriction tables*.

For $G = \mathbb{Z}/4$ and $H = \mathbb{Z}/2$ the induction-restriction table is:

	χ_+	χ_-
χ_1	1	0
χ_2	1	0
χ_3	0	1
χ_4	0	1

For $G = S_3$ and $H = \{1, (12)\}$ the induction-restriction table is:

	χ_+	χ_-
χ_1	1	0
χ_2	0	1
χ_3	1	1

In both cases, the rows give the decompositions of $\text{Res}_H^G \chi_i$ and the columns give the decompositions of $\text{Ind}_H^G \chi_{\pm}$.

3.2. Induced representations. Last time we proved that the induced character $\text{Ind}_H^G \chi_V$ is the character of some representation which is uniquely determined up to isomorphism. Today I want to construct that representation explicitly. There are two methods, abstract and concrete. The abstract version is short and immediately implies Frobenius reciprocity. The concrete version is complicated but you can see what the representation actually looks like.

Definition 3.7. *If V is a representation of H and $H \leq G$ then the induced representation is defined to be*

$$\text{Ind}_H^G V = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$$

3.2.1. Frobenius reciprocity. One of the main theorems follows immediately from basic properties of tensor product:

Theorem 3.8 (Frobenius reciprocity). *If V is a representation of H and W is a representation of G then*

$$\text{Hom}_G(\text{Ind}_H^G V, W) \cong \text{Hom}_H(V, \text{Res}_H^G W)$$

This follows from:

Theorem 3.9 (adjunction formula).

$$\text{Hom}_R({}_R M_S \otimes_S {}_S V, {}_R N) \cong \text{Hom}_S({}_S V, \text{Hom}_R(M_S, N))$$

And the easy formula:

$$\text{Hom}_R(R, N) \cong N$$

Letting $M = R$ and $S \subseteq R$, we get the following.

Corollary 3.10. *If S is a subring of R , V is an S -module and W is an R -module then*

$$\text{Hom}_R(R \otimes_S V, W) \cong \text{Hom}_S(V, W)$$

Putting $R = \mathbb{C}[G]$, $S = \mathbb{C}[H]$, this gives Frobenius reciprocity. Thus it suffices to prove the adjunction formula.

Proof of adjunction formula. The first step follow from the definition of the tensor product. When S is a noncommutative ring, such as $S = \mathbb{C}[H]$, the tensor product $M \otimes_S V$ is sometimes called a “balanced product.” It is an abelian group characterized by the following universal property:

- (1) There is a mapping $f : M \times V \rightarrow M \otimes_S V$ which is

- (a) *bilinear* in the sense that it is a homomorphism in each variable ($f(x, -) : V \rightarrow M \otimes_S V$ and $f(-, v) : M \rightarrow M \otimes_S V$ are homomorphisms for all x, v) and
- (b) f is *balanced* in the sense that

$$f(xs, v) = f(x, sv)$$

for all $x \in M, s \in S, v \in V$.

In other words, f is bilinear and balanced.

- (2) For any other bilinear, balanced mapping $g : M \times V \rightarrow W$ there is a unique homomorphism $\widehat{g} : M \otimes_S V \rightarrow W$ so that $g = \widehat{g} \circ f$

Let $\text{BiLin}(M \times_S V, W)$ denote the set of all balanced bilinear maps $M \times V \rightarrow W$. Then the universal property says that

$$\text{BiLin}(M \times_S V, W) \cong \text{Hom}(M \otimes_S V, W)$$

On the other hand the definitions of balanced and bilinear imply that

$$\text{BiLin}(M \times_S V, W) \cong \text{Hom}_S(V, \text{Hom}(M, W))$$

The balanced bilinear map $\phi : M \times_S V \rightarrow W$ corresponds to its adjoint $\widehat{\phi} : V \rightarrow \text{Hom}(M, W)$ given by $\widehat{\phi}(v)(x) = \phi(x, v)$.

- (1) $\phi(x, v)$ is linear in x iff $\widehat{\phi}(v) \in \text{Hom}(M, W)$. This is clear.
- (2) $\phi(x, v)$ is linear in v iff $\widehat{\phi}$ is linear, i.e., gives a homomorphism of abelian groups $V \rightarrow \text{Hom}(M, W)$. This is also clear.
- (3) Finally, ϕ is balanced iff $\phi(xs, v) = \phi(x, sv)$ iff

$$\widehat{\phi}(sv)(x) = \widehat{\phi}(v)(xs) = [s\widehat{\phi}(v)](x)$$

$$\text{iff } \widehat{\phi}s = s\widehat{\phi}.$$

In the case when M is an R - S -bimodule we just need to observe the obvious fact that ϕ is an R -homomorphism in the first coordinate iff $\widehat{\phi}(V) \subseteq \text{Hom}_R(M, W)$.

The adjunction formula follows from these observations. □

3.2.2. example. Here is the simplest example of an induced representation. Take $G = \mathbb{Z}/4 = \{1, \tau, \tau^2, \tau^3\}$ and $H = \mathbb{Z}/2 = \{1, \sigma\}$ where $\sigma = \tau^2$. Let ρ be the one dimensional sign representation $\rho(\sigma) = -1$. Let V denote the H -module of the representation. So, $H = \mathbb{C}$ with σ acting by -1 .

What is the induced representation $\text{Ind}_{\mathbb{Z}/2}^{\mathbb{Z}/4} \rho$?

The induced module is $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$ which is 2-dimensional. It is generated by four elements $1 \otimes 1, \tau \otimes 1, \tau^2 \otimes 1, \tau^3 \otimes 1$. But $\tau^2 = \sigma$. So,

$$\tau^2 \otimes 1 = 1 \otimes \sigma 1 = -1 \otimes 1$$

and

$$\tau^3 \otimes 1 = \tau \otimes \sigma 1 = -\tau \otimes 1$$

So, $\mathbb{C}[G] \otimes V$ is two dimensional with basis $w_1 = 1 \otimes 1, w_2 = \tau \otimes 1$ and τ acts by: $\tau w_1 = w_2$ and $\tau w_2 = \tau^2 \otimes 1 = -1 \otimes 1 = -w_1$. So, the matrix of the representation $\text{Ind}_H^G \rho = \phi$ is given by:

$$\phi(\tau) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Since G is cyclic this determines the other matrices:

$$\phi(\tau^2) = \phi(\tau)^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \phi(\tau^3) = \phi(\tau)^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Notice that matrices are all “monomial” which means that they have exactly one nonzero entry in every row and every column. The induced representation is always given by monomial matrices.

3.2.3. monomial matrices. A *monomial matrix* of size m with coefficients in a group H is defined to be an element of $\text{Mat}_m(\mathbb{Z}[H])$ having exactly one nonzero entry in every row and every column and so that those entries lie in H . Every monomial matrix M is a product of a permutation matrix P_σ and a diagonal matrix D :

$$M = P_\sigma D(h_1, h_2, \dots, h_m)$$

Here P_σ is the matrix obtained from the identity matrix I_m by permuting the rows by the permutation σ . For example, if $\sigma = (132)$ then

$$P_{(132)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

This is obtained by taking the identity matrix, moving row 1 which is $(1, 0, 0)$ to row $\sigma(1) = 3$, moving row 2 which is $(0, 1, 0)$ to row $\sigma(2) = 1$, etc. The entries of the matrix are:

$$(P_\sigma)_{ij} = \begin{cases} 1 & \text{if } i = \sigma(j) \\ 0 & \text{otherwise} \end{cases}$$

The notation for the diagonal group is the obvious one: $D(h_1, \dots, h_m)$ is the diagonal matrix with (i, i) entry h_i . So, for example,

$$P_{(132)} D(h_1, h_2, h_3) = \begin{pmatrix} 0 & h_2 & 0 \\ 0 & 0 & h_3 \\ h_1 & 0 & 0 \end{pmatrix}$$

So, h_j is in the j th column.

How do monomial matrices multiply? We need to calculate:

$$P_\sigma D(h_1, \dots, h_m) P_\tau D(\ell_1, \dots, \ell_m)$$

But

$$D(h_1, \dots, h_m) P_\tau = P_\tau D(h_{\tau(1)}, \dots, h_{\tau(m)})$$

So,

$$(3.1) \quad P_\sigma D(h_1, \dots, h_m) P_\tau D(\ell_1, \dots, \ell_m) = P_{\sigma\tau} D(h_{\tau(1)}\ell_1, \dots, h_{\tau(m)}\ell_m)$$

Definition 3.11. Let $M_m(H)$ denote the group of all $m \times m$ monomial matrices with coefficients in H . We denote the elements by

$$M(\sigma; h_1, \dots, h_m) = P_\sigma D(h_1, \dots, h_m)$$

3.2.4. *monomial representation.* Suppose that H is a subgroup of a group G with index $|G : H| = m$. Then

$$G = t_1H \cup t_2H \cup \dots \cup t_mH$$

where t_1, \dots, t_m form what is called a (left) *transversal* which is a set of representatives for the left cosets of H . Then we will get a *monomial representation* by which I mean a homomorphism

$$\rho : G \rightarrow M_m(H)$$

First, I start with the *permutation representation*

$$\pi : G \rightarrow S_m$$

which is given by the action of G on the set of left cosets of H . If $\sigma \in G$ then

$$\sigma t_j H = t_i H$$

where $i = \sigma(j) = \pi(\sigma)(j)$.

For example, suppose $G = S_3, H = \{1, (12)\}$. Choose the transversal: $t_1 = 1, t_2 = (13), t_3 = (23)$. Then $\sigma = (13)$ acts on the three left cosets by transposing the first two and fixing the third:

$$(13)t_1H = t_2H, \quad (13)t_2H = t_1H, \quad (13)t_3H = t_3H$$

Therefore, $\pi(13) = (12)$.

Now, look at the element of H that we get:

$$\sigma t_j = t_{\sigma(j)} h_j$$

where

$$\boxed{h_j = t_{\sigma(j)}^{-1} \sigma t_j}$$

Definition 3.12. *The monomial representation*

$$\rho : G \rightarrow M_m(H)$$

is given by

$$\rho(\sigma) = M(\pi(\sigma); t_{\sigma(1)}^{-1}\sigma t_1, \dots, t_{\sigma(m)}^{-1}\sigma t_m)$$

The following calculation verifies that ρ is a homomorphism:

$$\begin{aligned} \rho(\sigma)\rho(\tau) &= M(\pi(\sigma); t_{\sigma(1)}^{-1}\sigma t_1, \dots, t_{\sigma(m)}^{-1}\sigma t_m)M(\pi(\tau); t_{\tau(1)}^{-1}\tau t_1, \dots, t_{\tau(m)}^{-1}\tau t_m) \\ &= M(\pi(\sigma)\pi(\tau); \dots, \left(t_{\sigma(i)}^{-1}\sigma t_i\right) \left(t_{\tau(j)}^{-1}\tau t_j\right), \dots) \end{aligned}$$

But $i = \tau(j)$ by the formula (3.1). So,

$$\left(t_{\sigma(i)}^{-1}\sigma t_i\right) \left(t_{\tau(j)}^{-1}\tau t_j\right) = t_{\sigma\tau(j)}^{-1}\sigma\tau t_j$$

and

$$\rho(\sigma)\rho(\tau) = M(\pi(\sigma\tau); \dots, t_{\sigma\tau(j)}^{-1}\sigma\tau t_j, \dots) = \rho(\sigma\tau)$$

3.2.5. induced representation as monomial representation. Suppose that $\phi : H \rightarrow GL(k, \mathbb{C})$ is a k -dimensional representation of H and $V \cong \mathbb{C}^k$ is the corresponding H -module. Then I claim that the induced representation $\text{Ind}_H^G \phi$ is a monomial representation. More precisely the statement is:

Proposition 3.13. *The induced representation*

$$\psi = \text{Ind}_H^G \phi : G \rightarrow GL(mk, \mathbb{C})$$

is the composition of the monomial representation $\rho : G \rightarrow M_m(H)$ with the homomorphism

$$M_m(\phi) : M_m(H) \rightarrow M_m(GL(k, \mathbb{C})) \subseteq GL(mk, \mathbb{C})$$

induced by $\phi : H \rightarrow GL(k, \mathbb{C})$.

Proof. As a right H -module, $\mathbb{C}[G]$ is free of rank m with a basis given by a left transversal t_1, \dots, t_m . So,

$$\mathbb{C}[G] \cong t_1\mathbb{C}[H] \oplus \dots \oplus t_m\mathbb{C}[H]$$

As a G -module the induced representation is defined to be

$$\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V = (t_1 \otimes V) \oplus \dots \oplus (t_m \otimes V)$$

An arbitrary element is given by $\sum_j t_j \otimes v_j$ where v_j are arbitrary elements of V . Each $\sigma \in G$ acts by

$$\sigma \sum t_j \otimes v_j = \sum \sigma t_j \otimes v_j = \sum t_{\sigma(j)} h_j \otimes v_j = \sum t_{\sigma(j)} \otimes \phi(h_j) v_j$$

In other words, σ acts on V^m by multiplying the j th copy of V by the matrix

$$\phi(h_j) = \phi(t_{\sigma(j)}^{-1} \sigma t_j)$$

and then moving it to the $\sigma(j)$ slot. So:

$$\text{Ind}_H^G \phi = M(\pi(\sigma); \dots, \phi(t_{\sigma(j)}^{-1} \sigma t_j), \dots)$$

This is $M_m(\phi)$ applied to the standard monomial representation as I claimed. \square

Proposition 3.14. *The character of the induced representation is the induced character.*

Proof. This is a simple calculation. The trace of a monomial matrix is given by the points left fixed by the permutation representation $\pi(\sigma)$:

$$\begin{aligned} \text{Tr}(\text{Ind}_H^G \phi) &= \text{Tr} M(\pi(\sigma); \dots, \phi(t_{\sigma(j)}^{-1} \sigma t_j), \dots) \\ &= \sum_{j=\sigma(j)} \text{Tr} \phi(t_{\sigma(j)}^{-1} \sigma t_j) = \sum_{j=1}^m \chi_\phi(t_j^{-1} \sigma t_j) \end{aligned}$$

because $\chi_\phi(t_j^{-1} \sigma t_j) = 0$ when $j \neq \sigma(j)$.

Since χ_ϕ is a class function on H ,

$$\chi_\phi(t_j^{-1} \sigma t_j) = \chi_\phi(h^{-1} t_j^{-1} \sigma t_j h)$$

for all $h \in H$. So,

$$\text{Tr}(\text{Ind}_H^G \phi) = \frac{1}{|H|} \sum_{h \in H} \sum_{j=1}^m \chi_\phi(h^{-1} t_j^{-1} \sigma t_j h)$$

Since $t_j h$ runs over all the elements of G , this is equal to

$$\text{Ind}_H^G \chi_\phi(\sigma) = \frac{1}{|H|} \sum_{\tau \in G} \chi_\phi(\tau^{-1} \sigma \tau)$$

proving the proposition. \square

3.3. Artin's theorem. One of the main theorems is that all characters on finite groups are integer linear combinations of characters induced from abelian subgroups. I don't have time to do this theorem. But I can prove a weaker version which says that all characters are rational linear combinations of characters induced from cyclic subgroups.

Before I prove this, I want to make sense out of the statement of the theorem. What happens when we take linear combinations of characters when the coefficients are arbitrary integers or rational numbers?

3.3.1. character ring.

Definition 3.15. The character ring $R(G)$ of G is defined to be the ring of all virtual characters which are defined to be differences of effective characters:

$$f = \chi_V - \chi_W$$

These can also be described as integer linear combination of irreducible characters:

$$f = \sum n_i \chi_i, \quad n_i \in \mathbb{Z}$$

$R(G)$ is a ring because (pointwise) sums and products of effective characters are effective. So, the same holds for virtual characters.

Proposition 3.16. A group homomorphism $\phi : H \rightarrow G$ induces a ring homomorphism $\phi^* : R(G) \rightarrow R(H)$. In particular, if $H \leq G$,

$$\text{Res}_H^G : R(G) \rightarrow R(H)$$

is a ring homomorphism.

I won't prove this because it is sort of obvious and I don't need it. I want to look at the induction map.

Proposition 3.17. If $H \leq G$ then

$$\text{Ind}_H^G : R(H) \rightarrow R(G)$$

is a group homomorphism, i.e., it is additive.

Proof. This follows from the fact that tensor product distributes over direct sum:

$$\begin{aligned} \text{Ind}_H^G(V \oplus W) &= \mathbb{C}[G] \otimes_{\mathbb{C}[H]} (V \oplus W) \\ &\cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V \oplus \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W \\ &= \text{Ind}_H^G V \oplus \text{Ind}_H^G W \end{aligned}$$

□

3.3.2. *statement of the theorem.* We want a collection of subgroups $\mathcal{X} = \{H\}$ of G with the property that the maps $\text{Ind}_H^G : R(H) \rightarrow R(G)$ taken together for all $H \in \mathcal{X}$ give an epimorphism

$$\sum \text{Ind}_H^G : \bigoplus_{H \in \mathcal{X}} R(H) \rightarrow R(G)$$

This would say that every (effective) character on G is an integer linear combination of characters induced from the subgroups $H \in \mathcal{X}$. But we will only get this rationally which is the same as saying that the cokernel is a finite group.

Theorem 3.18 (Artin). *Suppose that \mathcal{X} is a collection of subgroups $H \leq G$. Then the following conditions are equivalent.*

- (1) $\forall \sigma \in G \exists H \in \mathcal{X}$ so that H contains a conjugate of σ .
- (2) Every character on G is a rational linear combination of characters induced from the subgroups $H \in \mathcal{X}$.

As an example, the collection of cyclic subgroups of G satisfies condition (1) since every element of G is contained in a cyclic subgroup.

3.3.3. *example: D_4 .* Take the dihedral group

$$G = D_4 = \{1, \sigma, \sigma^2, \sigma^3, \tau, \tau\sigma, \tau\sigma^2, \tau\sigma^3\}$$

Let $\mathcal{X} = \{\mathbb{Z}/4, \langle \tau \rangle, \langle \tau\sigma \rangle\}$. These three subgroups meet all of the conjugacy classes of D_4 . So, Artin's theorem applies. To find the image of the induction map we start with the character table of D_4 :

	1	σ^2	σ	τ	$\sigma\tau$
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	-1	-1	1
χ_4	1	1	1	-1	-1
χ_5	2	-2	0	0	0

From this we can easily compute the induction-restriction table:

D_4	$\mathbb{Z}/4$				$\langle \tau \rangle$	$\langle \tau\sigma \rangle$	
	χ_1	χ_i	χ_{-1}	χ_{-i}	χ_+	χ_-	χ_+
χ_1	1				1		1
χ_2			1		1		1
χ_3			1			1	1
χ_4	1					1	1
χ_5		1		1	1	1	1

Here χ_ξ denotes the one dimensional character of a cyclic group of order n which sends the generator to ξ (which must be an n th root of unity).

This 5×8 matrix T gives the induction map:

$$\begin{aligned} R(\mathbb{Z}/4) \oplus R(\langle \tau \rangle) \oplus R(\langle \tau\sigma \rangle) &\xrightarrow{\text{Ind}} R(D_4) \\ \mathbb{Z}^4 \oplus \mathbb{Z}^2 \oplus \mathbb{Z}^2 &\xrightarrow{\text{multiplication by } T} \mathbb{Z}^5 \end{aligned}$$

Artin's theorem says that the cokernel of this map is a finite group. To find this group we use integer row and column operations, which change the basis for \mathbb{Z}^5 and \mathbb{Z}^8 respectively, to reduce the matrix T to the form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \end{pmatrix}$$

This means that the cokernel of the induction map is $\mathbb{Z}/2$. So, for any representation V of D_4 , twice the character of V is a sum of virtual characters induced from virtual representations of the three cyclic subgroup in the list \mathcal{X} .

3.3.4. *proof of the theorem.* (2) \Rightarrow (1). Let $\sigma \in G$. Then there is an irreducible character χ_i so that $\chi_i(\sigma) \neq 0$. Since χ_i is a rational linear combination of induced characters from $H \in \mathcal{X}$, there must be some $H \in \mathcal{X}$ and some representation V of H so that $\text{Ind}_H^G \chi_V(\sigma) \neq 0$. By the definition of induced character this implies that some conjugate of σ lies in H .

(1) \Rightarrow (2). Suppose that (2) is false. Then the set of induced virtual characters forms a subgroup L of $R(G) \cong \mathbb{Z}^b$ of rank $a < b$. Let ϕ_1, \dots, ϕ_a be a set of characters induced from elements $H \in \mathcal{X}$ which span L . We can decompose each ϕ_i into an integer linear combination of the irreducible characters χ_j :

$$\phi_i = \sum n_{ij} \chi_j$$

The numbers n_{ij} form an $a \times b$ matrix which defines a \mathbb{Q} linear map:

$$(n_{ij}) : \mathbb{Q}^b \rightarrow \mathbb{Q}^a$$

Since $a < b$ this linear map has a kernel, i.e., there are rational numbers c_j not all zero so that

$$\sum_j n_{ij} c_j = 0 \quad \forall i$$

Multiplying by the denominators, we may assume the numbers c_j are integers. This gives a nonzero virtual character

$$\sum c_j \chi_j = \chi_V - \chi_W$$

which is orthogonal to all the ϕ_i and therefore all $\phi \in L$:

$$\langle \phi_i, \chi_W - \chi_{W'} \rangle = \left\langle \phi_i, \sum c_j \chi_j \right\rangle = \sum n_{ij} c_j = 0$$

But L contains all induced characters:

$$\phi = \text{Ind}_H^G V$$

for all $H \in \mathcal{X}$ and all representations V of H . So, by Frobenius reciprocity, we have:

$$\langle \phi, \chi_W - \chi_{W'} \rangle = \langle \text{Ind}_H^G V, \chi_W - \chi_{W'} \rangle_G = \langle V, \text{Res}_H^G(\chi_W - \chi_{W'}) \rangle_H = 0$$

Since this is true for all representations V of H , we must have

$$\text{Res}_H^G(\chi_W - \chi_{W'}) = 0$$

for all $H \in \mathcal{X}$. This in turn implies that

$$\chi_W(\sigma) = \chi_{W'}(\sigma)$$

for all $\sigma \in H$.

But, for any $\sigma \in G$ there is an $H \in \mathcal{X}$ which contains a conjugate of σ . But then

$$\chi_W(\sigma) = \chi_{W'}(\sigma)$$

So, the virtual character $\chi_W - \chi_{W'}$ must be zero, which is a contradiction. This proves the theorem. \square