MATH 101B: HOMEWORK

1. Homework 01

The following three problems are due next Thursday (1/25/7).

1.1. Show that the category of finite abelian groups contains no non-trivial projective or injective objects. (Use the Fundamental Theorem: all finite abelian groups are direct sums of cyclic $p$-groups $\mathbb{Z}/p^n$.)

For any abelian group $A$, $\text{Hom}(\mathbb{Z}/n, A)$ is isomorphic to the set of all $a \in A$ so that $na = 0$ with the isomorphism even by evaluating functions on a fixed generator of $\mathbb{Z}/n$. If $n = p^k$ ($p$ prime) and $j \geq 0$ this gives

$$\text{Hom}(\mathbb{Z}/p^k, \mathbb{Z}/p^j) \cong p^j \mathbb{Z}/p^j \cong \mathbb{Z}/p^k$$

So, the short exact sequence

$$(1.1) \quad 0 \rightarrow \mathbb{Z}/p^k \xrightarrow{\alpha} \mathbb{Z}/p^{2k} \xrightarrow{\beta} \mathbb{Z}/p^k \rightarrow 0$$

gives the exact sequence

$$0 \rightarrow \text{Hom}(\mathbb{Z}/p^k, \mathbb{Z}/p^k) \xrightarrow{\alpha^\sharp} \text{Hom}(\mathbb{Z}/p^k, \mathbb{Z}/p^{2k}) \xrightarrow{\beta^\sharp} \text{Hom}(\mathbb{Z}/p^k, \mathbb{Z}/p^k)$$

which is isomorphic

$$0 \rightarrow \mathbb{Z}/p^k \xrightarrow{\alpha^\sharp} \mathbb{Z}/p^k \xrightarrow{\beta^\sharp} \mathbb{Z}/p^k$$

Therefore, $\alpha^\sharp$ is an isomorphism and $\beta^\sharp = 0$. This means that $\mathbb{Z}/p^k$ is not projective for any $k > 0$. This implies that there are no nontrivial projectives in the category of finite abelian groups since any such projective $P$ would have a direct summand of the form $\mathbb{Z}/p^k$ for $k > 0$ by the fundamental theorem of finite abelian groups and any direct summand of a projective module is projective. (To see that suppose that $P = X \oplus Y$ is projective and $g : A \rightarrow B$ is an epimorphism. Then any morphism $f : X \rightarrow B$ extends to a morphism $f + 0 : P = X \oplus Y \rightarrow B$ which lifts to $A$. The restriction of the lifting to $X$ gives a lifting of $f$ making $X$ projective.)

Similarly, if we apply $\text{Hom}(\cdot, \mathbb{Z}/p^k)$ to the exact sequence (1.1) we get:

$$0 \rightarrow \text{Hom}(\mathbb{Z}/p^k, \mathbb{Z}/p^k) \xrightarrow{\beta^\sharp} \text{Hom}(\mathbb{Z}/p^{2k}, \mathbb{Z}/p^k) \xrightarrow{\alpha^\sharp} \text{Hom}(\mathbb{Z}/p^{2k}, \mathbb{Z}/p^k)$$

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which is isomorphic to

$$0 \to \mathbb{Z}/p^k \xrightarrow{\beta} \mathbb{Z}/p^k \xrightarrow{\alpha} \mathbb{Z}/p^k$$

So, $\alpha^\sharp = 0$ which implies that $\mathbb{Z}/p^k$ is not injective for $k > 0$. Therefore there is no nontrivial injective object in the category of finite abelian groups.

1.2. Show that abelian categories have push-outs and pull-backs. I.e., Given an abelian category $\mathcal{A}$ and morphisms $f : A \to C, g : B \to C$ there exists an object $D$ (called the pull-back) with morphisms $\alpha : D \to A, \beta : D \to B$ forming a commuting square ($f \circ \alpha = g \circ \beta$) so that for any other object $X$ with maps to $A, B$ forming another commuting square, there exists a unique morphism $X \to D$ making a big commutative diagram. (Draw the diagram.) Hint: There is an exact sequence

$$0 \to D \to A \oplus B \to C$$

I will use the notation $j_A, j_B$ for the inclusion maps of $A, B$ into $A \oplus B$ and $p_A, p_B$ for the projections to $A, B$. Then $j_A \circ p_A + j_B \circ p_B$ as part of the definition of a direct sum (or as a consequence as explained in the notes).

Since $A \oplus B$ is the coproduct of $A$ and $B$, the morphisms $f$ and $-g$ induce a morphism

$$h = (f, -g) : A \oplus B \to C$$

so that $h \circ j_A = f, h \circ j_B = -g$. Let $D \to A \oplus B$ be the kernel of this homomorphism. In other words $D$ is the kernel in the exact sequence

$$0 \to D \xrightarrow{k} A \oplus B \xrightarrow{h} C$$

Let $\alpha = p_A \circ k : D \to A$ and $\beta = p_B \circ k : D \to B$ which is written $k = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. Then

**Claim 1.** $f \circ \alpha = g \circ \beta : D \to C$. In other words, $f \circ \alpha - g \circ \beta = 0$. This is a calculation:

$$f \circ \alpha - g \circ \beta = (h \circ j_A) \circ (p_A \circ k) + (h \circ j_B) \circ (p_B \circ k)$$

$$= h \circ (j_A \circ p_A + j_B \circ p_B) \circ k = h \circ k = 0$$

This calculation is often written in the matrix notation:

$$h \circ k = (f, -g) \circ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = f \circ \alpha - g \circ \beta$$

**Claim 2.** For any object $X$ and any pair of morphisms $\alpha' : X \to A, \beta' : X \to B$ so that $f \circ \alpha' = g \circ \beta'$ there is a unique morphism $\gamma : X \to D$ so that $\alpha' = \alpha \circ \gamma$ and $\beta' = \beta \circ \gamma$. 

For existence, let \( \phi = \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} : X \to A \oplus B \) be the product of \( \alpha' \) and \( \beta' \). I.e., the unique morphism so that \( \alpha' = p_A \circ \phi \) and \( \beta' = p_B \circ \phi \). Then
\[
\begin{align*}
  h \circ \phi &= h \circ (j_A \circ p_A + j_B \circ p_B) \circ \phi = f \circ \alpha' - g \circ \beta' = 0
\end{align*}
\]
Therefore, there is a unique morphism \( \gamma : X \to D \) so that \( h \circ \gamma = \phi \).
Then
\[
\begin{align*}
  \alpha' &= p_A \circ \phi = p_A \circ h \circ \gamma = \alpha \circ \gamma
\end{align*}
\]
and similarly, \( \beta' = \beta \circ \gamma \).

For uniqueness suppose that \( \gamma' : X \to D \) is another morphism so that \( \alpha' = \alpha \circ \gamma' \) and \( \beta' = \beta \circ \gamma' \). Then \( \phi' = h \circ \gamma' \) is another morphism so that \( p_A \circ \phi' = \alpha' \) and \( p_B \circ \phi' = \beta' \). By the universal property, \( \phi' = \phi \) which implies that \( \gamma' = \gamma \).

The conclusion is that \( D \) is the pull-back.

If we reserve all the arrow in the above argument (or equivalently, repeat the same argument in the opposite category \( A^{op} \)) we see that the push-out of a diagram
\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{s} & & \downarrow{} \\
C & &
\end{array}
\]
is the cokernel of
\[
\begin{pmatrix}
  f \\
  -g
\end{pmatrix} : A \to B \oplus C
\]

1.3. Let \( k \) be a field and let \( R \) be the polynomial ring \( R = k[X] \). Let \( Q \) be the \( k \) vector space of all sequences:
\[
(a_0, a_1, a_2, \cdots)
\]
where \( a_i \in k \) with the action of \( X \in R \) given by shifting to the left:
\[
X(a_0, a_1, a_2, \cdots) = (a_1, a_2, a_3, \cdots)
\]
Then show that \( Q \) is injective. Hint: first prove that any homomorphism \( f : A \to Q \) is determined by its first coordinate.

Let \( p_i : Q \to k \) be the projection to the \( i \)-th coordinate. This is linear map (i.e., a morphism of \( k \)-modules) and not a morphism of \( R \)-modules.

**Lemma 1.1.** For any \( R \)-module \( A \) there is an isomorphism
\[
\phi : \text{Hom}_R(A, Q) \cong \text{Hom}_k(A, k)
\]
given by sending \( f : A \to Q \) to \( \phi(f) = p_1 \circ f \).
Proof. \( f \) can be reconstructed from \( g = \phi(f) \) as follows. For any \( a \in A \), let
\[
h(a) = (g(a), g(Xa), g(X^2a), g(X^3a), \cdots)
\]
If \( a \) is replaced by \( Xa \) then this entire sequence shifts to the left. Therefore \( h(Xa) = Xh(a) \) making \( h \) into a homomorphism of \( R \)-modules \( h : A \to Q \). To see that \( f = h \) look at the coordinates of \( f(a) \)
\[
f(a) = (f_0(a), f_1(a), f_2(a), \cdots)
\]
g\((a)\) is the first coordinate: \( g(a) = f_0(a) \) and the other coordinates are given by shifting to the left and then taking the first coordinate:
\[
f_n(a) = p_0(X^nf(a)) = p_0(f(X^na)) = g(X^na) = h_n(a)
\]
Also, if we start with \( g \) and take \( h \) then \( \phi(h) = g \). So, \( \phi \) if a bijection and thus an isomorphism. \( \square \)

To show that \( Q \) is injective as an \( R \)-module suppose that
\[
(1.2) \quad 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0
\]
is a short exact sequence of \( R \)-modules. Then we want to show that induced sequence
\[
(1.3) \quad 0 \to \text{Hom}_R(C, Q) \xrightarrow{\beta^t} \text{Hom}_R(B, Q) \xrightarrow{\alpha^t} \text{Hom}_R(A, Q) \to 0
\]
is exact. By composing with \( p_0 \) we get the sequence
\[
(1.4) \quad 0 \to \text{Hom}_k(C, k) \xrightarrow{\beta^t} \text{Hom}_k(B, k) \xrightarrow{\alpha^t} \text{Hom}_k(A, k) \to 0
\]
which is exact since (1.2) is a split exact sequence of \( k \). Therefore the isomorphic sequence (1.3) is also exact making \( Q \) into an injective \( R \)-module.