MATH 101B: HOMEWORK

1. Homework 01

The following three problems are due next Thursday (1/25/7).

1.1. Show that the category of finite abelian groups contains no nontrivial projective or injective objects. (Use the Fundamental Theorem: all finite abelian groups are direct sums of cyclic p-groups $\mathbb{Z}/p^n$.)

For any abelian group A, $\mathrm{Hom}(\mathbb{Z}/n, A)$ is isomorphic to the set of all $a \in A$ so that $na = 0$ with the isomorphism even by evaluating functions on a fixed generator of $\mathbb{Z}/n$. If $n = p^k$ (p prime) and $j \geq 0$ this gives

$$\mathrm{Hom}(\mathbb{Z}/p^k, \mathbb{Z}/p^{j+k}) \cong p^j\mathbb{Z}/p^{j+k}\mathbb{Z} \cong \mathbb{Z}/p^k$$

So, the short exact sequence

$$0 \to \mathbb{Z}/p^k \xrightarrow{\alpha} \mathbb{Z}/p^{2k} \xrightarrow{\beta} \mathbb{Z}/p^k \to 0 \tag{1.1}$$

gives the exact sequence

$$0 \to \mathrm{Hom}(\mathbb{Z}/p^k, \mathbb{Z}/p^k) \xrightarrow{\alpha_\sharp} \mathrm{Hom}(\mathbb{Z}/p^k, \mathbb{Z}/p^{2k}) \xrightarrow{\beta_\sharp} \mathrm{Hom}(\mathbb{Z}/p^k, \mathbb{Z}/p^k)$$

which is isomorphic

$$0 \to \mathbb{Z}/p^k \xrightarrow{\alpha_\sharp} \mathbb{Z}/p^k \xrightarrow{\beta_\sharp} \mathbb{Z}/p^k$$

Therefore, $\alpha_\sharp$ is an isomorphism and $\beta_\sharp = 0$. This means that $\mathbb{Z}/p^k$ is not projective for any $k > 0$. This implies that there are no nontrivial projectives in the category of finite abelian groups since any such projective P would have a direct summand of the form $\mathbb{Z}/p^k$ for $k > 0$ by the fundamental theorem of finite abelian groups and any direct summand of a projective module is projective. (To see that suppose that $P = X \oplus Y$ is projective and $g : A \to B$ is an epimorphism. Then any morphism $f : X \to B$ extends to a morphism $f + 0 : P = X \oplus Y \to B$ which lifts to A. The restriction of the lifting to X gives a lifting of f making X projective.)

Similarly, if we apply $\mathrm{Hom}(-, \mathbb{Z}/p^k)$ to the exact seqence (1.1) we get:

$$0 \to \mathrm{Hom}(\mathbb{Z}/p^k, \mathbb{Z}/p^k) \xrightarrow{\beta^\sharp} \mathrm{Hom}(\mathbb{Z}/p^{2k}, \mathbb{Z}/p^k) \xrightarrow{\alpha^\sharp} \mathrm{Hom}(\mathbb{Z}/p^{2k}, \mathbb{Z}/p^k)$$

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which is isomorphic to

$$0 \to \mathbb{Z}/p^k \xrightarrow{\beta^\sharp} \mathbb{Z}/p^k \xrightarrow{\alpha^\sharp} \mathbb{Z}/p^k$$

So, $\alpha^\sharp = 0$ which implies that $\mathbb{Z}/p^k$ is not injective for $k > 0$. Therefore there is no nontrivial injective object in the category of finite abelian groups.

1.2. Show that abelian categories have push-outs and pull-backs. I.e., Given an abelian category $\mathcal{A}$ and morphisms $f : A \to C, g : B \to C$ there exists an object D (called the *pull-back*) with morphisms $\alpha : D \to A, \beta : D \to B$ forming a commuting square ($f \circ \alpha = g \circ \beta$) so that for any other object X with maps to A, B forming another commuting square, there exists a unique morphism $X \to D$ making a big commutative diagram. (Draw the diagram.) Hint: There is an exact sequence

$$0 \to D \to A \oplus B \to C$$

I will use the notation j_A, j_B for the inclusion maps of A, B into $A \oplus B$ and p_A, p_B for the projections to A, B. Then $j_A \circ p_A + j_B \circ p_B$ as part of the definition of a direct sum (or as a consequence as explained in the notes).

Since $A \oplus B$ is the coproduct of A and B, the morphisms f and $-g$ induce a morphism

$$h = (f, -g) : A \oplus B \to C$$

so that $h \circ j_A = f, h \circ j_B = -g$ Let $D \to A \oplus B$ be the kernel of this homomorphism. In other words D is the kernel in the exact sequence

$$0 \to D \xrightarrow{k} A \oplus B \xrightarrow{h} C$$

Let $\alpha = p_A \circ k : D \to A$ and $\beta = p_B \circ k : D \to B$ which is written $k = \binom{\alpha}{\beta}$. Then

<u>Claim 1</u>. $f \circ \alpha = g \circ \beta : D \to C$. In other words, $f \circ \alpha - g \circ \beta = 0$. This is a calculation:

$$\begin{aligned} f \circ \alpha - g \circ \beta &= (h \circ j_A) \circ (p_A \circ k) + (h \circ j_B) \circ (p_B \circ k) \\ &= h \circ (j_A \circ p_A + j_B \circ p_B) \circ k = h \circ k = 0 \end{aligned}$$

This calculation is often written in the matrix notation:

$$h \circ k = (f, -g) \circ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = f \circ \alpha - g \circ \beta$$

<u>Claim 2</u>. For any object X and any pair of morphisms $\alpha' : X \to A, \beta' : X \to B$ so that $f \circ \alpha' = g \circ \beta'$ there is a unique morphism $\gamma : X \to D$ so that $\alpha' = \alpha \circ \gamma$ and $\beta' = \beta \circ \gamma$.

For existence, let $\phi = \binom{\alpha'}{\beta'} : X \to A \oplus B$ be the product of α' and β'. I.e., the unique morphism so that $\alpha' = p_A \circ \phi$ and $\beta' = p_B \circ \phi$. Then

$$h \circ \phi = h \circ (j_A \circ p_A + j_B \circ p_B) \circ \phi = f \circ \alpha' - g \circ \beta' = 0$$

Therefore, there is a unique morphism $\gamma : X \to D$ so that $h \circ \gamma = \phi$. Then

$$\alpha' = p_A \circ \phi = p_A \circ h \circ \gamma = \alpha \circ \gamma$$

and similarly, $\beta' = \beta \circ \gamma$.

For uniqueness suppose that $\gamma' : X \to D$ is another morphism so that $\alpha' = \alpha \circ \gamma'$ and $\beta' = \beta \circ \gamma'$. Then $\phi' = h \circ \gamma'$ is another morphism so that $p_A \circ \phi' = \alpha'$ and $p_B \circ \phi' = \beta'$. By the universal property, $\phi' = \phi$ which implies that $\gamma' = \gamma$.

The conclusion is that D is the pull-back.

If we reserve all the arrow in the above argument (or equivalently, repeat the same argument in the opposite category $\mathcal{A}^{op}$) we see that the push-out of a diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \\ C & & \end{array}$$

is the cokernel of

$$\begin{pmatrix} f \\ -g \end{pmatrix} : A \to B \oplus C$$

1.3. Let k be a field and let R be the polynomial ring $R = k[X]$. Let Q be the k vector space of all sequences:

$$(a_0, a_1, a_2, \cdots)$$

where $a_i \in k$ with the action of $X \in R$ given by shifting to the left:

$$X(a_0, a_1, a_2, \cdots) = (a_1, a_2, a_3, \cdots)$$

Then show that Q is injective. Hint: first prove that any homomorphism $f : A \to Q$ is determined by its first coordinate.

Let $p_i : Q \to k$ be the projection to the i-th coordinate. This is linear map (i.e., a morphism of k-modules) and not a morphism of R-modules.

Lemma 1.1. *For any R-module A there is an isomorphism*

$$\phi : \mathrm{Hom}_R(A, Q) \cong \mathrm{Hom}_k(A, k)$$

given by sending $f : A \to Q$ to $\phi(f) = p_1 \circ f$.

Proof. f can be reconstructed from $g = \phi(f)$ as follows. For any $a \in A$, let

$$h(a) = (g(a), g(Xa), g(X^2 a), g(X^3 a), \cdots)$$

If a is replaced by Xa then this entire sequence shifts to the left. Therefore $h(Xa) = Xh(a)$ making h into a homomorphism of R-modules $h : A \to Q$. To see that $f = h$ look at the coordinates of $f(a)$

$$f(a) = (f_0(a), f_1(a), f_2(a), \cdots)$$

$g(a)$ is the first coordinate: $g(a) = f_0(a)$ and the other coordinates are given by shifting to the left and then taking the first coordinate:

$$f_n(a) = p_0(X^n f(a)) = p_0(f(X^n a) = g(X^n a) = h_n(a)$$

Also, if we start with g and take h then $\phi(h) = g$. So, ϕ if a bijection and thus an isomorphism. □

To show that Q is injective as an R-module suppose that

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \tag{1.2}$$

is a short exact sequence of R-modules. Then we want to show that induced sequence

$$0 \to \operatorname{Hom}_R(C, Q) \xrightarrow{\beta^\sharp} \operatorname{Hom}_R(B, Q) \xrightarrow{\alpha^\sharp} \operatorname{Hom}_R(A, Q) \to 0 \tag{1.3}$$

is exact. By composing with p_0 we get the sequence

$$0 \to \operatorname{Hom}_k(C, k) \xrightarrow{\beta^\sharp} \operatorname{Hom}_k(B, k) \xrightarrow{\alpha^\sharp} \operatorname{Hom}_k(A, k) \to 0 \tag{1.4}$$

which is exact since (1.2) is a split exact sequence of k. Therefore the isomorphic sequence (1.3) is also exact making Q into an injective R-module.