

MATH 101B: HOMEWORK

2. HOMEWORK 02

The following three problems are due next Thursday (2/2/7). The strict deadline is 1:30pm Friday.

2.1. Give a precise description of the ring R indicated below and show that it has the property that left R -modules are the same as chain complexes of abelian groups with three terms, i.e.,

$$C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0$$

where C_1, C_2, C_3 are abelian groups and d_1, d_2 are homomorphisms so that $d_1 \circ d_2 = 0$ and a homomorphism of R -modules is a chain map $f : C \rightarrow D$, i.e. it consists of three homomorphisms $f_i : C_i \rightarrow D_i$ so that the following diagram commutes.

$$\begin{array}{ccccc} C_2 & \longrightarrow & C_1 & \longrightarrow & C_0 \\ \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 \\ D_2 & \longrightarrow & D_1 & \longrightarrow & D_0 \end{array}$$

R is a quotient ring:

$$R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} & \mathbb{Z} \\ 0 & 0 & \mathbb{Z} \end{pmatrix} / \begin{pmatrix} 0 & 0 & \mathbb{Z} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

[Hint: R is additively free abelian with five generators, three of which are idempotents e_0, e_1, e_2 and $C_i = e_i C$.]

R is additively free abelian with five generators, $e_0, e_1, e_2, \alpha, \beta$ where the first three are diagonal matrices

$$e_0 = \text{diag}(1, 0, 0), \quad e_1 = \text{diag}(0, 1, 0), \quad e_2 = \text{diag}(0, 0, 1)$$

and α is a matrix with a 1 in the $(0, 1)$ -entry and 0's elsewhere and β is a matrix with 1 in the $(1, 2)$ -entry and zeros elsewhere.

Given an R -module C the corresponding chain complex is given as follows. Let $C_i = e_i C$. Since $1 = e_0 + e_1 + e_2$ and e_i are orthogonal idempotents in the sense that $e_i e_j = \delta_{ij} e_j$, we get a decomposition

$$C = e_0 C \oplus e_1 C \oplus e_2 C = C_0 \oplus C_1 \oplus C_2$$

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Since $\alpha = e_0 \alpha e_1$, multiplication by α is zero on $C_0 \oplus C_2$ and has image in C_0 . So it gives a homomorphism $C_1 \rightarrow C_0$. Similarly, multiplication by β gives a homomorphism $C_2 \rightarrow C_1$. Since $\alpha\beta = 0$, the composition of these two homomorphisms is zero. Therefore, we have a chain complex structure on C .

Conversely, suppose we have a chain complex $C_2 \xrightarrow{d} C_1 \xrightarrow{d} C_0$. Then we have an action of R on $C = \oplus C_i$ by

$$\begin{pmatrix} a_{00} & a_{01} & 0 \\ 0 & a_{11} & a_{12} \\ 0 & 0 & a_{22} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{00}x_0 + a_{01}d(x_1) \\ a_{11}x_1 + a_{12}d(x_2) \\ a_{22}x_2 \end{pmatrix}$$

where we write elements of C in column matrix form. We see that multiplication by e_i is projection onto C_i and multiplication by the matrix α sends (x_0, x_1, x_2) to $(d(x_1), 0, 0)$, i.e., it is projection to C_1 followed by $d : C_1 \rightarrow C_0$ followed by inclusion $C_0 \rightarrow C$. Similarly for β . So, the chain complex corresponding to this R -module is the original chain complex.

Conversely, if we start with an R -module, form the chain complex and then make the chain complex back into an R -module by the above construction then we get back that same R -module. This is because $d_1 : C_1 \rightarrow C_0$ is α restricted to C_1 and $d_2 : C_2 \rightarrow C_1$ is β restricted to C_2 .

2.2. Describe the chain complex corresponding to the free R -module R^n (n finite).

First take $n = 1$. Then R is free with 3 generators: $e_0, e_1, e_2, \alpha, \beta$. Multiplying on the left with e_0 kills all but e_0, α . So, $e_0R \cong \mathbb{Z}^2$ with free generators e_0, α . Similarly, $e_1R \cong \mathbb{Z}^2$ with free generators e_1, β and $e_2R \cong \mathbb{Z}$ with free generator e_2 . So, the chain complex is:

$$\mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^2 \xrightarrow{d_1} \mathbb{Z}^2$$

As we already discussed, d_2 is left multiplication by β which sends the generator e_2 of $C_2 = \mathbb{Z}$ to the second generator β of $C_1 = \mathbb{Z}^2$. And d_1 is left multiplication by α which sends the first generator e_1 of C_1 to the second generator α of $C_0 \cong \mathbb{Z}^2$ and sends the second generator β of C_1 to zero.

If we take the direct sum of n copies of this we get the chain complex

$$\mathbb{Z}^n \xrightarrow{d_2} \mathbb{Z}^{2n} \xrightarrow{d_1} \mathbb{Z}^{2n}$$

where d_2 is the inclusion into the first factor of $\mathbb{Z}^{2n} = \mathbb{Z}^n \oplus \mathbb{Z}^n$ and d_1 is the projection to the second coordinate of C_1 followed by inclusion

into the second factor of C_0 . In matrix notation:

$$d_2 = \begin{pmatrix} 0 \\ I_n \end{pmatrix}, \quad d_1 = \begin{pmatrix} 0 & I_n \\ 0 & 0 \end{pmatrix}$$

2.3. Assume without proof that the analogous statements hold for right R -modules, namely they are cochain complexes

$$C^2 \leftarrow C^1 \leftarrow C^0$$

where $C^i = Ce_i$ (just as $C_i = e_iC$). Give a description (as a chain complex with three terms) of the injective R -module $\text{Hom}_{\mathbb{Z}}(R_R^n, \mathbb{Q}/\mathbb{Z})$.

The right R -module R is the cochain complex $C^2 = Re_2 \cong \mathbb{Z}^2$ with free generators e_2, β , $C^1 = Re_1 \cong \mathbb{Z}^2$ with generators e_1, α and $C^0 = Re_0 \cong \mathbb{Z}$ with generator e_0 . $d^1 : C^1 \rightarrow C^2$, being right multiplication by β , is an isomorphism of the first factor of C^1 to the second factor of C^2 and $d^0 : C^0 \rightarrow C^1$ is an isomorphism of C^0 onto the second factor of C^1 .

If we hom into \mathbb{Q}/\mathbb{Z} we get

$$(\mathbb{Q}/\mathbb{Z})^2 \xrightarrow{d_2} (\mathbb{Q}/\mathbb{Z})^2 \xrightarrow{d_1} \mathbb{Q}/\mathbb{Z}$$

Where d_2 is an isomorphism from the second factor of C_2 to the first factor of C_1 and d_1 is an isomorphism from the second factor of C_1 to C_0 . $\text{Hom}_{\mathbb{Z}}(R_R^n, \mathbb{Q}/\mathbb{Z})$ is a direct sum of n copies of this chain complex. So, it is

$$(\mathbb{Q}/\mathbb{Z})^{2n} \xrightarrow{d_2} (\mathbb{Q}/\mathbb{Z})^{2n} \xrightarrow{d_1} (\mathbb{Q}/\mathbb{Z})^n$$

where

$$d_2 = \begin{pmatrix} 0 & I_n \\ 0 & 0 \end{pmatrix}, \quad d_1 = (0 \quad I_n)$$