MATH 101B: HOMEWORK

3. Homework 03 answers

The following problem was due Thursday (2/8/7).

Compute $\text{Ext}^i_{\mathbb{Q}[X]}(\mathbb{Q}[X]/(f), \mathbb{Q}[X]/(g))$ using both the projective resolution of $\mathbb{Q}[X]/(f)$ and the injective coresolution of $\mathbb{Q}[X]/(g)$.

[First take the projective resolution P_* of $\mathbb{Q}[X]/(f)$. Then

$$\text{Ext}^i_{\mathbb{Q}[X]}(\mathbb{Q}[X]/(f), \mathbb{Q}[X]/(g)) \cong H^i(\text{Hom}_{\mathbb{Q}[X]}(P_*, \mathbb{Q}[X]/(g)))$$

Then, find the injective (co)resolution Q_* of $\mathbb{Q}[X]/(g)$. The Ext groups can also be found by the formula

$$\text{Ext}^i_{\mathbb{Q}[X]}(\mathbb{Q}[X]/(f), \mathbb{Q}[X]/(g)) \cong H^i(\text{Hom}_{\mathbb{Q}[X]}(\mathbb{Q}[X]/(f), Q_*))$$

The theorem that says that these two definitions are equivalent, we don't have time to prove. But working out this example might give you an idea on why it is true for $i = 0, 1$.]

Actually, I did end up proving that theorem.

3.1. projective resolution.

Since $\mathbb{Q}[X]$ is a domain, the projective resolution of $\mathbb{Q}[X]/(f)$ is given by

$$0 \to \mathbb{Q}[X] \xrightarrow{f\cdot} \mathbb{Q}[X] \to \mathbb{Q}[X]/(f) \to 0$$

where $f\cdot$ is multiplication by f. Since $\text{Hom}_R(R, M) \cong M$, we get:

$$\text{Hom}_{\mathbb{Q}[X]}(\mathbb{Q}[X], \mathbb{Q}[X]/(g)) \cong \mathbb{Q}[X]/(g)$$

The isomorphism $\phi_x \leftrightarrow x$ is given by $\phi_x(1) = x$. The map induced by $f\cdot$ is precomposition with $f\cdot$ which corresponds to multiplication by f:

$$(f\cdot)^*(x) = (f\cdot)^\sharp(\phi_x) = \phi_x(f \cdot 1) = f \cdot x$$

So, $\text{Ext}^1_{\mathbb{Q}[X]}(\mathbb{Q}[X]/(f), \mathbb{Q}[X]/(g))$ is the cokernel of the map

$$\mathbb{Q}[X]/(g) \to \mathbb{Q}[X]/(g)$$

given by multiplication by f. This is $\mathbb{Q}[X]$ modulo the ideal I consisting of all $\mathbb{Q}[X]$-linear combinations of f and g. In other words, $I = (f, g)$. But $\mathbb{Q}[X]$ is PID. So this ideal is generated by one element, the greatest common divisor of f and g. Call this h. Then

$$\text{Ext}^1_{\mathbb{Q}[X]}(\mathbb{Q}[X]/(f), \mathbb{Q}[X]/(g)) \cong \mathbb{Q}[X]/(h)$$

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$\mathrm{Ext}^0_{\mathbb{Q}[X]}(\mathbb{Q}[X]/(f), \mathbb{Q}[X]/(g)) \cong \mathrm{Hom}_{\mathbb{Q}[X]}(\mathbb{Q}[X]/(f), \mathbb{Q}[X]/(g))$ is the kernel of the map

$$\mathbb{Q}[X]/(g) \xrightarrow{f\cdot} \mathbb{Q}[X]/(g)$$

This is $K/(g)$ where K is the kernel of the composite map

$$\mathbb{Q}[X] \to \mathbb{Q}[X]/(g) \xrightarrow{f\cdot} \mathbb{Q}[X]/(g)$$

But this means that K is the set of all polynomials which, when multiplied by f become divisible by g. To figure out what this is you need to write f and g as products $f = ha, g = hb$ where $h = gcd(f, g)$. (So, a, b are relatively prime.) $kf = kha$ is divisible by $g = hb$ iff k is divisible by b. Therefore, $K = (b)$ and

$$\mathrm{Ext}^0_{\mathbb{Q}[X]}(\mathbb{Q}[X]/(f), \mathbb{Q}[X]/(g)) \cong K/(g) \cong \mathbb{Q}[X]/(h)$$

where the isomorphism $\mathbb{Q}[X]/(h) \cong K/(g)$ is given by multiplication by b.

3.2. **Injective resolution.** To get an injective resolution of $\mathbb{Q}[X]/(g)$ we first need a lemma:

Lemma 3.1. *The module $\mathbb{Q}[X]/(g)$ is isomorphic to its dual:*

$$\mathbb{Q}[X]/(g)^\wedge = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}[X]/(g), \mathbb{Q}) \cong \mathbb{Q}[X]/(g)$$

Suppose that this is true. Then the short exact sequence:

$$0 \to \mathbb{Q}[X] \xrightarrow{g\cdot} \mathbb{Q}[X] \to \mathbb{Q}[X]/(g) \to 0 \tag{3.1}$$

when dualized gives an injective coresolution of $\mathbb{Q}[X]/(g) \cong \mathbb{Q}[X]/(g)^\wedge$:

$$0 \to \mathbb{Q}[X]/(g) \to \mathbb{Q}[X]^\wedge \xrightarrow{\cdot g} \mathbb{Q}[X]^\wedge \to 0 \tag{3.2}$$

This is exact since (3.1) splits over $\mathbb{Q}$. The injective module $\mathbb{Q}[X]$ consists of infinite sequences of rational numbers

$$(a_0, a_1, a_2, \cdots)$$

where X acts by moving the sequence to the left:

$$X(a_0, a_1, \cdots) = (a_1, a_2, \cdots)$$

From the previous homework we know that

$$\mathrm{Hom}_{\mathbb{Q}[X]}(\mathbb{Q}[X]/(f), \mathbb{Q}[X]^\wedge) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}[X]/(f), \mathbb{Q})$$

which, by the lemma, is isomorphic to $\mathbb{Q}[X]/(f)$.

When we apply $\mathrm{Hom}_{\mathbb{Q}[X]}(\mathbb{Q}[X]/(f), -)$ to the injective coresolution (3.2) we get

$$\mathbb{Q}[X]/(f) \xrightarrow{\cdot g} \mathbb{Q}[X]/(f)$$

which has cokernel

$$\mathrm{Ext}^1_{\mathbb{Q}[X]}(\mathbb{Q}[X]/(f), \mathbb{Q}[X]/(g)) = \mathbb{Q}[X]/(f,g) = \mathbb{Q}[X](h)$$

where h is the greatest common divisor of f and g and kernel

$$\mathrm{Ext}^0_{\mathbb{Q}[X]}(\mathbb{Q}[X]/(f), \mathbb{Q}[X]/(g)) = (a)/(f) \cong \mathbb{Q}[X]/(h)$$

where $a = f/h$.

Proof of the lemma. Suppose first that g is irreducible of degree n. Then $M = \mathbb{Q}[X]/(g)$ is n dimensional over $\mathbb{Q}$ and is annihilated by multiplication by g. Then $M^\wedge$ is also n dimensional and is annihilated by multiplication by g. So, $M^\wedge \cong M$. Next suppose that g is a power $g = p^k$ of an irreducible polynomial p of degree n. Then M is characterized by the fact that it is nk dimensional and is annihilated by p^k but not by p^{k-1}. But $M^\wedge$ also has this property. So, $M^\wedge \cong M$.

Finally suppose that $g = \prod p_i^{k_i}$ where p_i are distinct irreducible monic polynomials. Then $M = \bigoplus \mathbb{Q}[X]/(p_i^{k_i})$ and

$$M^\wedge \cong \bigoplus \mathbb{Q}[X]/(p_i^{k_i})^\wedge \cong \bigoplus \mathbb{Q}[X]/(p_i^{k_i}) = M$$

□