5. Homework 05 answers

The following problems were due Thursday (3/8/7).

(1) (Problems #3 on page 374) If $L \subseteq E$ are finitely generated field extension(s) of K then show that the transcendence degree of E over K is equal to the sum of the transcendence degree of E over L and the transcendence degree of L over K.

Lang says: Let $x_1, \cdots, x_n$ be a transcendence basis for L over K and let $y_1, \cdots, y_m$ be a transcendence basis for E over L. Then show that $x_1, \cdots, x_n$, $y_1, \cdots, y_m$ form a transcendence basis for E over K.

So, we need to show

(a) $x_1, \cdots, x_n, y_1, \cdots, y_m$ are algebraically independent and

(b) Every element of E is algebraic over $K(x_1, \cdots, y_m)$.

Proof of (1): Suppose that $f(X, Y) \in K[X, Y]$ so that

$$f(x_1, \cdots, x_n, y_1, \cdots, y_m) = 0$$

Rewrite $f(X, Y)$ as

$$f(X, Y) = \sum_\alpha f_\alpha(X) Y^\alpha$$

where $f_\alpha(X) \in K[X]$ for all multiindices α. If we substitute $X_i = x_i$ then each $f_\alpha(x) \in K[x] \subseteq L$. So, $f(x, Y)$ is an element of $L[Y]$. Since the y_j are algebraically independent over L and $f(x, y) = 0$ it must be that $f_\alpha(x) = 0$ for all α. But, the x_i are algebraically independent over K. So, $f_\alpha(X) = 0$ for all α. So, $f(X, Y)$ is the zero polynomial and the x_i and y_j together form an alg. ind. set.

Proof of (2): We have to show that E is algebraic over $K(x, y)$. So, take any element $a \in E$. Then a is algebraic over $L(y) \cong L(Y)$. So, there is a polynomial $g(Z)$ with coefficients in $L(y)$ so that $g(a) = 0$. But each of the coefficients of g is a rational function in $y_1, \cdots, y_m$ with coefficients in L. The set of these coefficients forms a finite set $S \subset L$ so that a is algebraic over $K(x, S, y)$. But each element of S is algebraic over $K(x)$ and therefore also over $K(x, y)$. So, a is also algebraic over $K(x, y)$ as claimed.

(2) Let $R = K[X, Y]/(f)$ where $f(X, Y) = (X - a)Y^2 - (X - b)$ for some $a \neq b \in K$. Find a transcendental element Z of R so that R is integral over $K[Z]$. [Use the proof of Noether Normalization.]

The proof of Noether normalization says that $Z = X - Y^m$ for some large m. But $m = 1$ is large enough. Then $Z = X - Y$ and $X = Z + Y$. We see that $K[X, Y] = K[Z, Y]$. The polynomial f, when written in terms of Y and Z becomes:

$$f(Z, Y) = (Z + Y - a)Y^2 - (Z - b) = Y^3 + (Z - a)Y^2 - Y - (Z - b)$$

Since this is a monic polynomial in Y with coefficients in $K[Z]$ we see two things:

(a) Y is integral over $K[Z]$

(b) $R = K[Z, Y]/(f)$ is a free module over $K[Z]$ of rank 3.

This implies that R is infinite dimensional as a vector space over K. Therefore, it cannot be an algebraic extension of K. So, Z is transcendental and R is integral over $K[Z]$.