6. Homework 06 answers

The following problems was due Thursday (3/22/7).

Assume R, M are both Noetherian.

(1) Show that for any ideal I in R there are only finitely many minimal primes containing I. [Take a maximal counterexample.]

Let $M = R/I$. Then $I \subseteq \mathfrak{p}$ iff $M_\mathfrak{p} \neq 0$. (Proof: $M_\mathfrak{p} = 0$ iff $1 \sim 0$ in $M_\mathfrak{p}$ iff $\exists s \notin \mathfrak{p}$ s.t. $s = 0$ in M iff $s \in I$.)

Therefore, the minimal primes containing I are exactly the minimal supporting primes of R/I which are the same as the minimal associated primes. We proved that there are only finitely many of these.

(2) Suppose that $\mathfrak{p}$ is a prime and $n > 0$. Let

$$\mathfrak{p}^{(n)}M := \mathfrak{p}^n M_\mathfrak{p} | M$$

Show that this is a $\mathfrak{p}$-primary submodule of M.

Let $N = \mathfrak{p}^{(n)}M$. Then, by definition, N is the inverse image of $\mathfrak{p}^n M_\mathfrak{p}$ in M. So, M/N is isomorphic to an R-submodule of $M_\mathfrak{p}/\mathfrak{p}^n M_\mathfrak{p}$. So, it suffices to show that $\mathfrak{p}$ is the only associated prime of $M_\mathfrak{p}/\mathfrak{p}^n M_\mathfrak{p}$.

In $M_\mathfrak{p}$ each element $s \in S = R\backslash\mathfrak{p}$ act as isomorphisms (with inverse $s^{-1} \in R_\mathfrak{p}$). This means that on $M_\mathfrak{p}$ and more generally on any $R_\mathfrak{p}$-module, when considered as an R-module, the annihilator of any nonzero element will be disjoint from S, i.e., $ann_R(x) \subseteq \mathfrak{p}$. On the other hand $\mathfrak{p}$ annihilates the entire module $\mathfrak{p}^k M_\mathfrak{p}/\mathfrak{p}^{k+1}M_\mathfrak{p}$. So, every nonzero element x will have $ann_R(x) = \mathfrak{p}$. This means that $\mathfrak{p}$ is the only associated prime (in R) of $Q_k = \mathfrak{p}^k M_\mathfrak{p}/\mathfrak{p}^{k+1}M_\mathfrak{p}$. But, $Q = M_\mathfrak{p}/\mathfrak{p}^n M_\mathfrak{p}$ is an extension of the quotients Q_k. So, $\mathfrak{p}$ is the only associated prime of Q. Since M/N is a submodule of Q, $\mathfrak{p}$ is also the only prime associated to M/N. So, N is $\mathfrak{p}$-primary.

(3) If $\phi : R \to S$ is a homomorphism of Noetherian rings and M is an S-module then show that

$$\text{ass}_R(M) = \phi^*(\text{ass}_S(M))$$

where $\phi^* : Spec(S) \to Spec(R)$ is the map induced by ϕ. ($\phi^*(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$)

Suppose first that M has only one associated prime in S, call it $\mathfrak{p}$. Then $\mathfrak{p}$ is the set of zero divisors of M and every zero divisor acts nilpotently on M in the sense that a power of it annihilates M. Then $\mathfrak{q} = \phi^*(\mathfrak{p})$ is the set of zero divisors of M in R. And each element of $\mathfrak{q}$ acts nilpotently on M. This means that $\text{ass}_R(M) = \{\mathfrak{q}\}$. So, the theorem is true in this case.

Let $0 = \bigcap Q_i$ be a primary decomposition of $0 \subseteq M$ as an S-submodule. Suppose that Q_i is $\mathfrak{p}_i$-primary. Then

$$\text{ass}_S(M/Q_i) = \{\mathfrak{p}_i\} \Rightarrow \text{ass}_R(M/Q_i) = \{\mathfrak{q}_i\}$$

where $\mathfrak{q}_i = \phi^*(\mathfrak{p}_i)$ by the argument above. This implies that $0 = \bigcap Q_i$ is also a primary decomposition of 0 as an R-submodule of M. So,

$$\text{ass}_R(M) = \bigcup \text{ass}_R(M/Q_i) = \{\mathfrak{q}_1, \cdots, \mathfrak{q}_n\} = \phi^*(\text{ass}_S(M))$$