The following problems was due Thursday (3/22/7).

Assume $R, M$ are both Noetherian.

1. **Show that for any ideal $I$ in $R$ there are only finitely many minimal primes containing $I$.** [Take a maximal counterexample.]

   Let $M = R/I$. Then $I \subseteq \mathfrak{p}$ iff $M_{\mathfrak{p}} \neq 0$. (Proof: $M_{\mathfrak{p}} = 0$ iff $1 \sim 0$ in $M_{\mathfrak{p}}$ iff $\exists s \notin \mathfrak{p}$ s.t. $s = 0$ in $M$ iff $s \in I$.)

   Therefore, the minimal primes containing $I$ are exactly the minimal supporting primes of $R/I$ which are the same as the minimal associated primes. We proved that there are only finitely many of these.

2. **Suppose that $\mathfrak{p}$ is a prime and $n > 0$. Let**

   $$\mathfrak{p}^{(n)}M := \mathfrak{p}^nM|_M$$

   **Show that this is a $\mathfrak{p}$-primary submodule of $M$.**

   Let $N = \mathfrak{p}^{(n)}M$. Then, by definition, $N$ is the inverse image of $\mathfrak{p}^nM_{\mathfrak{p}}$ in $M$. So, $M/N$ is isomorphic to an $R$-submodule of $M_{\mathfrak{p}}/\mathfrak{p}^nM_{\mathfrak{p}}$. So, it suffices to show that $\mathfrak{p}$ is the only associated prime of $M_{\mathfrak{p}}/\mathfrak{p}^nM_{\mathfrak{p}}$.

   In $M_{\mathfrak{p}}$ each element $s \in S = R\setminus \mathfrak{p}$ act as isomorphisms (with inverse $s^{-1} \in R_{\mathfrak{p}}$). This means that on $M_{\mathfrak{p}}$ and more generally on any $R_{\mathfrak{p}}$-module, when considered as an $R$-module, the annihilator of any nonzero element will be disjoint from $S$, i.e., $ann_R(x) \subseteq \mathfrak{p}$. On the other hand $\mathfrak{p}$ annihilates the entire module $\mathfrak{p}^kM_{\mathfrak{p}}/\mathfrak{p}^{k+1}M_{\mathfrak{p}}$. So, every nonzero element $x$ will have $ann_R(x) = \mathfrak{p}$. This means that $\mathfrak{p}$ is the only associated prime (in $R$) of $Q_k = \mathfrak{p}^kM_{\mathfrak{p}}/\mathfrak{p}^{k+1}M_{\mathfrak{p}}$. But, $Q = M_{\mathfrak{p}}/\mathfrak{p}^nM_{\mathfrak{p}}$ is an extension of the quotients $Q_k$. So, $\mathfrak{p}$ is the only associated prime of $Q$. Since $M/N$ is a submodule of $Q$, $\mathfrak{p}$ is also the only prime associated to $M/N$. So, $N$ is $\mathfrak{p}$-primary.

3. **If $\phi : R \to S$ is a homomorphism of Noetherian rings and $M$ is an $S$-module then show that**

   $$\text{ass}_R(M) = \phi^*(\text{ass}_S(M))$$

   where $\phi^* : \text{Spec}(S) \to \text{Spec}(R)$ is the map induced by $\phi$. ($\phi^*(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$)

   Suppose first that $M$ has only one associated prime in $S$, call it $\mathfrak{p}$. Then $\mathfrak{p}$ is the set of zero divisors of $M$ and every zero divisor acts nilpotently on $M$ in the sense that a power of it annihilates $M$. Then $\mathfrak{q} = \phi^*(\mathfrak{p})$ is the set of zero divisors of $M$ in $R$. And each element of $\mathfrak{q}$ acts nilpotently on $M$. This means that $\text{ass}_R(M) = \{\mathfrak{q}\}$. So, the theorem is true in this case.

   Let $0 = \bigcap Q_i$ be a primary decomposition of $0 \subseteq M$ as an $S$-submodule. Suppose that $Q_i$ is $\mathfrak{p}_i$-primary. Then

   $$\text{ass}_S(M/Q_i) = \{\mathfrak{p}_i\} \Rightarrow \text{ass}_R(M/Q_i) = \{\mathfrak{q}_i\}$$

   where $\mathfrak{q}_i = \phi^*(\mathfrak{p}_i)$ by the argument above. This implies that $0 = \bigcap Q_i$ is also a primary decomposition of $0$ as an $R$-submodule of $M$. So,

   $$\text{ass}_R(M) = \bigcup \text{ass}_R(M/Q_i) = \{\mathfrak{q}_1, \cdots, \mathfrak{q}_n\} = \phi^*(\text{ass}_S(M))$$