

7. HOMEWORK 07 ANSWERS

The following problems were due Thursday (4/12/7).

(1) Show that $\mathbb{Z}[1/p]/\mathbb{Z}$ is an Artinian \mathbb{Z} -module. [Show that every proper subgroup is finite.]

The logic of the argument is as follows. This abelian group $A = \mathbb{Z}[1/p]/\mathbb{Z}$ has a sequence of elements a_1, a_2, \dots satisfying the following two conditions:

- (a) The sequence generates the group.
- (b) If a subgroup $B \leq A$ does not contain a_{n+1} then it has at most p^n elements.

Together, these two statements imply that any proper subgroup of A is finite. So, the DCC must hold for subgroups and A is Artinian.

(a) The generating elements are: $a_n = 1/p^n$. By definition, any element of $\mathbb{Z}[1/p]$ is an integer linear combination of these elements. So, they generate $\mathbb{Z}[1/p]$ and, consequently, they also generate A .

(b) Suppose that B is a subgroup of A which does not contain $1/p^{n+1}$. Then the highest denominator in the elements of B is p^n . So, the only possible elements of B are $i/p^n + \mathbb{Z}$ where $0 \leq i < p^n$. So, $|B| \leq p^n$.

(2) Show that the center of the ring $Mat_n(R)$ is isomorphic to the center of R . [Show that the center $Z(Mat_n(R))$ consists of the scalar multiples aI_n of the identity matrix where $a \in Z(R)$.]

Let $Z = Z(Mat_n(R))$ be the center of $Mat_n(R)$. Let x_{ij} be the matrix with 1 in the (i, j) position and zero everywhere else. Then, any matrix in $Mat_n(R)$ can be written as $A = \sum a_{ij}x_{ij}$ where $a_{ij} \in R$. If $A \in Z$ then, e.g.,

$$x_{12}A = \sum_{j=1}^n a_{2j}x_{1j} = Ax_{12} = \sum_{i=1}^n a_{i1}x_{i2}$$

Comparing the coefficients of x_{12} on both sides we see that $a_{22} = a_{11}$. For $j \neq 2$ the RHS has no x_{1j} terms. So, $a_{2j} = 0$. Changing 1, 2 to other indices we see that $a_{ij} = 0$ if $i \neq j$ and the diagonal terms a_{ii} must all be equal.

To see that these diagonal entries must lie in the center of R , take any $r \in R$. Then

$$(rx_{11})A = ra_{11}x_{11} = A(rx_{11}) = a_{11}rx_{11}$$

So, $ra_{11} = a_{11}r$.

(3) Suppose that M is both Artinian and Noetherian. Then show that

$$(1) \ r^n M = 0 \text{ for some } n > 0 \ (r^n M := \underbrace{rr \cdots r}_n M)$$

Since M is Noetherian, it has at least one maximal submodule (assuming $M \neq 0$). Therefore, the radical of M is a proper submodule. Repeating this process we get a descending sequence of submodules

$$M \supsetneq rM \supsetneq r^2M \supsetneq \cdots$$

Since M is Artinian, the sequence stops: $r^n M = r^{n+1}M$. But this can happen only if $r^n M = 0$

(2) $r^i M / r^{i+1}M$ is f.g. semisimple for all $i \geq 0$.

By the finite intersection lemma, $rM = \bigcap N_i$ is a finite intersection of maximal submodules N_i . So, M/rM is a submodule of the semisimple module $\bigoplus M/N_i$, making it semisimple. Similarly each of the other subquotients $r^i M / r^{i+1}M$ is semisimple.

(4) Prove the converse, i.e., (a) and (b) imply that M is both Artinian and Noetherian.

Suppose that $n = 1$. Then $rM = 0$ and $M \cong M/rM$ is semisimple by assumption. But we know that semisimple modules are both Artinian and Noetherian.

Now suppose that $n \geq 2$. We may assume by induction that rM is Artinian and Noetherian. We also know that M/rM satisfies both chain conditions. Let N_1, N_2, \dots be a sequence of submodules which is either increasing or decreasing. Then $N_i \cap rM$ must eventually become stationary. Similarly, the image of N_i in M/rM must also become stationary. Using the 5-lemma on the following diagram (just as we did for Noetherian modules over commutative rings), we see that the inclusion $N_i \hookrightarrow N_j$ is an isomorphism and thus an equality for sufficiently large i, j . ($i < j$ or $i > j$ depending on whether N_i is an ascending or descending chain.)

$$\begin{array}{ccccccc} 0 & \longrightarrow & N_i \cap rM & \longrightarrow & N_i & \longrightarrow & \frac{N_i + rM}{rM} \longrightarrow 0 \\ & & \downarrow = & & \downarrow \subseteq & & \downarrow = \\ 0 & \longrightarrow & N_j \cap rM & \longrightarrow & N_j & \longrightarrow & \frac{N_j + rM}{rM} \longrightarrow 0 \end{array}$$