8. Homework 08 answers

The following problems were due Thursday (4/19/7).

8.1. Let K be any field, let G be any finite group. Let $V = K$ be the trivial representation of G. Let $E = K[G]$ be the group ring considered as a representation (this is called the "regular representation") Find all G-homomorphisms

$$f : V \to E$$

(and show that you have a complete list).

Let $N \in K[G]$ denote the sum of all the elements of the group:

$$N = \sum_{\sigma \in G} \sigma$$

Then N is clearly invariant under multiplication by elements of G:

$$\tau N = N \quad \forall \tau \in G$$

For any $a \in K$ let $f_a : V \to E$ be given by $f_a(x) = axN$. Then this is a linear map which is G-equivariant:

$$f_a(\sigma x) = f_a(x) = axN = ax\sigma N = \sigma f_a(x)$$

To show that the f_a are the homomorphisms $V \to E$, let $f : V \to E$ be a homomorphism of G-modules. Let

$$f(1) = \sum a_\sigma \sigma$$

Then

$$f(1) = f(\tau 1) = \tau f(1) = \sum a_\sigma \tau\sigma$$

Comparing the coefficients of $\tau\sigma$ we see that $a_{\tau\sigma} = a_\sigma$ for all $\sigma, \tau \in G$. Taking $\sigma = 1$ we see that $a_\tau = a_1$ for all τ, i.e., the coefficients are the same. So, $f(1) = aN$ for some $a \in K$. But then $f(x) = axN$. So, $f = f_a$.

8.2. If K is the field with two elements and G is the group with two elements then show that $K[G]$ is not semisimple. [Using your answer to question 1, show that you have a short exact sequence $0 \to V \to E \to V \to 0$ which does not split.]

By problem 1 there are only two homomorphisms $V \to E$, namely $f_0 = 0$ and $f_1 : V \to E$ which has image $V_0 = \{0, N\}$. The second point is that all one dimensional representations are isomorphic to the trivial representation. This is because $\mathrm{Aut}_K(K) = K^\times$ has only one element. If E were semisimple, then $E = V_0 \oplus W$ where $W \subseteq E$ is another 1-dimensional submodule of E. But then $W \cong V$ and we would get another homomorphism $V \to E$ contradicting Problem 1. So, E is not a semisimple module. So, $K[G]$ is not a semisimple ring.