9. Homework 09 answers

The following problems were due Thursday (4/26/7).

9.1. Calculate the character table of the quaternion group Q and find the 2-dimensional irreducible representation.

If we mod out the central element t then we get $Q/\langle t\rangle \cong \mathbb{Z}/2\times\mathbb{Z}/2$ with character table

	1	i	j	k
χ_1	1	1	1	1
χ_2	1	−1	1	−1
χ_3	1	−1	−1	1
χ_4	1	1	−1	−1

Pulling this back to Q gives:

	1	i	j	k	t
χ_1	1	1	1	1	1
χ_2	1	−1	1	−1	1
χ_3	1	−1	−1	1	1
χ_4	1	1	−1	−1	1
χ_5	2	0	0	0	−2

Since characters determine the representation, the unique 2-dimensional irreducible representation is any representation with character χ_5. The module is the quaternions $\mathbb{H}$ viewed as a right $\mathbb{C}$ module. The units $\pm 1, \pm i, \pm j, \pm k$ in $\mathbb{H}$ form a group isomorphic to the quaternion group Q and left multiplication by these elements commutes with right multiplication by elements of $\mathbb{C}$ and is therefore $\mathbb{C}$-linear. To find matrices for the elements of Q we choose a basis for $\mathbb{H}$, say $v_1 = 1, v_2 = j$. Then the matrices for i, j, k are given by:

$$\rho(i) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \rho(j) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho(k) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

for example, $iv_1 = v_1 i, iv_2 = v_2(-i)$ making $\rho(i) = D(i, -i)$. Taking traces we see that this is the representation with character χ_5. So, it is the one we want.