

10. HOMEWORK 10

This was the last assignment. It was due Thursday (5/3/7).

10.1. If V, S are irreducible representations of G and S is one-dimensional, then show that $V \otimes S$ is also irreducible.

There are two proofs. The first proof uses the fact that tensor product distributes over direct sum and one dimensional characters have inverses. The second proof is a computation.

First note that $V \otimes \mathbb{C} \cong V$ as G -modules if \mathbb{C} is the trivial representation. This follows from the computation:

$$\chi_{V \otimes \mathbb{C}}(\sigma) = \chi_V(\sigma)\chi_1(\sigma) = \chi_V(\sigma) \cdot 1 = \chi_V(\sigma)$$

If χ is a one dimension character then $\bar{\chi}$ is its inverse since

$$\bar{\chi}(\sigma) = \bar{\zeta} = \zeta^{-1}$$

for all $\sigma \in G$. Consequently

$$(V \otimes S) \otimes \bar{S} \cong V \otimes S \otimes \bar{S} \cong V \otimes \mathbb{C} \cong V$$

is irreducible. This implies that $V \otimes S$ must be irreducible.

The second proof is just a computation. If $V \otimes S \cong \bigoplus n_i S_i$ then

$$\begin{aligned} \|\chi_{V \otimes S}\|^2 &= \sum n_i^2 = \langle \chi_V \chi_S, \chi_V \chi_S \rangle \\ &= \frac{1}{n} \sum \chi_V(\sigma) \chi_S(\sigma) \overline{\chi_V(\sigma) \chi_S(\sigma)} = \frac{1}{n} \sum \chi_V(\sigma) \overline{\chi_V(\sigma)} = 1 \end{aligned}$$

Which implies that $V \otimes S$ is irreducible.

10.2. Show that the irreducible representations of a product $G \times H$ are the tensor products of irreducible representations of G, H .

Suppose that ϕ_1, \dots, ϕ_a are the irreducible representations of G and ψ_1, \dots, ψ_b are the irreducible representations of H . Let $\pi_G : G \times H \rightarrow G, \pi_H : G \times H \rightarrow H$ be the projection maps.

Let $\tilde{\phi}_i = \phi_i \circ \pi_G, \tilde{\psi}_j = \psi_j \circ \pi_H$ be the pull-backs of ϕ_i, ψ_j to $G \times H$.

Claim 1: $\tilde{\phi}_i \otimes \tilde{\psi}_j$ is irreducible.

Proof: The restriction of $\tilde{\phi}_i \otimes \tilde{\psi}_j$ to $G = G \times 1$ is irreducible:

$$\text{Res}_G^{G \times H} \tilde{\phi}_i \otimes \tilde{\psi}_j = \phi_i$$

Claim 2: The irreducible representations $\tilde{\phi}_i \otimes \tilde{\psi}_j$ are distinct.

Proof: $\tilde{\phi}_i \otimes \tilde{\psi}_j$ restricts to ϕ_i on G and ψ_j on H .

Claim 3: $G \times H$ has ab conjugacy classes.

Proof: (σ, τ) and $(\sigma'\tau')$ are conjugate in $G \times H$ iff σ, σ' are conjugate in G and τ, τ' are conjugate in H .

These three together show that $\tilde{\phi}_i \otimes \tilde{\psi}_j$ give a complete set of irreducible representations of $G \times H$.

Another proof uses the equation:

$$\langle \chi_{\tilde{\phi}_i \otimes \tilde{\psi}_j}, \chi_{\tilde{\phi}_k \otimes \tilde{\psi}_\ell} \rangle = \langle \chi_{\tilde{\phi}_i}, \chi_{\tilde{\phi}_k} \rangle \langle \chi_{\tilde{\psi}_j}, \chi_{\tilde{\psi}_\ell} \rangle = \delta_{ik} \delta_{j\ell}$$

This equation shows simultaneously that $\tilde{\phi}_i \otimes \tilde{\psi}_j$ are irreducible and that they are distinct. Since there are ab of them, these are all the irreducible characters on $G \times H$.

10.3. Compute the character table of the alternating group A_4 and the induction-restriction table for the pair (S_4, A_4) .

First you need to find the conjugacy classes of A_4 . By the orbit-stabilizer formula, the size of the conjugacy class of σ in G is equal to the index of $C_G(\sigma)$ the stabilizer of σ in G . It is easy to see what these centralizers are:

σ	$C(\sigma)$	$ A_4 : C(\sigma) $
1	A_4	1
(123)	$\{1, (123), (132)\}$	4
(12)(34)	$\{1, (12)(34), (13)(24), (14)(23)\}$	3

Since A_4 has 8 elements of cycle type (abc) and (123) has only 4 conjugates, it must mean that there is another conjugacy class of 3-cycles which also has 4 elements (elements of $N \triangleleft G$ which are conjugate in G have the same number of conjugates in N). To find it you just conjugate (123) by any odd permutation, e.g., (23) to give (132).

The quotient of A_4 by the Klein 4-group $K = \{1, (12)(34), (13)(24), (14)(23)\}$ is cyclic of order 3. So, its character table is:

	1	(123)K	(132)K
χ_1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω

where $\omega = \frac{-1+\sqrt{3}}{2}$.

Pulling back to A_4 , we get:

	1	(123)	(132)	(12)(34)
χ_1	1	1	1	1
χ_2	1	ω	ω^2	1
χ_3	1	ω^2	ω	1
χ_4	3	0	0	-1

Here we use the fact that $(12)(34) \in K$ to get $\chi_i((12)(34)) = \chi_i(1)$ for $i = 1, 2, 3$ and $\chi_4(1) = 3$ is the only solution to the equation $12 = 1 + 1 + 1 + \chi_4(1)^2$. The other values of χ_4 are obtained by column orthogonality.

Looking at the character table of S_4 which I did in class we can get the induction restriction table:

	S_4					A_4			
	1	(12)	(123)	(12)(34)	(1234)	χ_1	χ_2	χ_3	χ_4
χ_1	1	1	1	1	1	1	0	0	0
χ_2	1	-1	1	1	-1	1	0	0	0
χ_3	2	0	-1	2	0	0	1	1	0
χ_4	3	1	0	-1	-1	0	0	0	1
χ_5	3	-1	0	-1	1	0	0	0	1

The induction-restriction table is obtained by looking at the values of the characters of S_4 on the conjugacy classes 1, (123), (12)(34). The restricted characters have the same value on the conjugacy classes (123) and (132).