2. Characters

If $\rho : G \to GL_d(\mathbb{C})$ is a representation of G over $\mathbb{C}$ then the *character* of ρ is the function

$$\chi_\rho : G \to \mathbb{C}$$

given by $\chi_\rho(\sigma) = \mathrm{Tr}(\rho(\sigma))$.

The main property of characters is that they determine the representation uniquely up to isomorphism. So, once we find all the characters (by constructing the character table) we will in some sense know all the representations. We will assume that all groups G are finite and all representations are finite dimensional over $\mathbb{C}$.

2.1. Basic properties.

The basic property of trace is that it is invariant under conjugation:

$$\mathrm{Tr}(ABA^{-1}) = \mathrm{Tr}(B)$$

Letting $A = \rho(\sigma), B = \rho(\tau)$ we get

$$\chi_\rho(\sigma\tau\sigma^{-1}) = \mathrm{Tr}(\rho(\sigma\tau\sigma^{-1})) = \mathrm{Tr}(\rho(\sigma)\rho(\tau)\rho(\sigma)^{-1}) = \mathrm{Tr}(\rho(\tau)) = \chi_\rho(\tau)$$

for any representation ρ. So:

Theorem 2.1. *Characters are class functions. (They have the same value on conjugate elements.)*

If $\rho : G \to \mathrm{Aut}_{\mathbb{C}}(V)$ is a representation of G over $\mathbb{C}$, then the *character of* ρ, also called the *character of* V, is defined to be the function

$$\chi_\rho = \chi_V : G \to \mathbb{C}$$

given by

$$\chi_V(\sigma) = \mathrm{Tr}(\rho(\sigma)) = \mathrm{Tr}(\phi \circ \rho(\sigma) \circ \phi^{-1})$$

for any linear isomorphism $\phi : V \xrightarrow{\approx} \mathbb{C}^d$.

There are three basic formulas that I want to explain. In order of difficulty they are:

(1) The character of a direct sum is the sum of the characters:

$$\chi_{V \oplus W} = \chi_V + \chi_W$$

(2) The character of a tensor product is the product of the characters:

$$\chi_{V \otimes W} = \chi_V \chi_W$$

(3) The character of the dual representation is the complex conjugate of the original character:

$$\chi_{V^*} = \overline{\chi_V}$$

2.1.1. *direct sum.* The trace of a direct sum of matrices is the sum of traces:

$$\mathrm{Tr}(A \oplus B) = \mathrm{Tr}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \mathrm{Tr}(A) + \mathrm{Tr}(B)$$

Theorem 2.2. *If V, W are two G-modules then*

$$\chi_{V \oplus W} = \chi_V + \chi_W$$

Proof. If $\rho_V, \rho_W, \rho_{V \oplus W}$ are the corresponding representations then

$$\rho_{V \oplus W}(\sigma) = \rho_V(\sigma) \oplus \rho_W(\sigma)$$

The theorem follows. □

2.1.2. *character formula using dual basis.* Instead of using traces of matrices, I prefer the following equivalent formula for characters using bases and dual bases.

If V is a G-module, we choose a basis $\{v_1, \cdots, v_d\}$ for V as a vector space over $\mathbb{C}$. Then recall that the *dual basis* for $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ consists of the dual vectors $v_1^*, \cdots, v_d^* : V \to \mathbb{C}$ given by

$$v_j^*\left(\sum_{i=1}^{d} a_i v_i\right) = a_j$$

I.e., v_j^* picks out the coefficient of v_j.

Proposition 2.3.

$$\chi_V(\sigma) = \sum_{i=1}^{d} v_i^*(\sigma v_i)$$

Proof. The matrix of the linear transformation $\rho(\sigma)$ has (i, j) entry $v_i^*(\sigma v_j)$ Therefore, its trace is $\sum v_i^*(\sigma v_i)$. □

For example, the trace of the identity map is

$$\mathrm{Tr}(id_V) = \sum_{i=1}^{d} v_i^*(v_i) = d$$

Theorem 2.4. *The value of the character at* 1 *is the dimension of the representation:*

$$\chi_V(1) = d = \dim_{\mathbb{C}}(V)$$

2.1.3. *tensor product.* If V, W are two G-modules then the *tensor product* $V \otimes W$ is defined to be the tensor product over $\mathbb{C}$ with the following action of G:

$$\sigma(v \otimes w) = \sigma v \otimes \sigma w$$

Theorem 2.5. *The character of $V \otimes W$ is the product of the characters of V and W. I.e.,*

$$\chi_{V \otimes W}(\sigma) = \chi_V(\sigma)\chi_W(\sigma)$$

for all $\sigma \in G$.

Proof. Choose bases $\{v_i\}, \{w_j\}$ for V, W with dual bases $\{v_i^*\}, \{w_j^*\}$. Then the tensor product $V \otimes W$ has basis elements $v_i \otimes w_j$ with dual basis elements $v_i^* \otimes w_j^*$. So, the character is:

$$\chi_{V \otimes W}(\sigma) = \sum_{i,j} (v_i^* \otimes w_j^*)\sigma(v_i \otimes w_j) = \sum_{i,j} (v_i^* \otimes w_j^*)(\sigma v_i \otimes \sigma w_j)$$

$$= \sum_{i,j} v_i^*(\sigma v_i) w_j^*(\sigma w_j) = \sum_i v_i^*(\sigma v_i) \sum_j w_j^*(\sigma w_j) = \chi_V(\sigma)\chi_W(\sigma)$$

□

2.1.4. *dual reprentation.* The dual space V^* is a right G-module. In order to make it a left G-module we have to invert the elements of the group. I.e., for all $f \in V^*$ we define

$$(\sigma f)(v) := f(\sigma^{-1} v)$$

Lemma 2.6.

$$\chi_{V^*}(\sigma) = \chi_V(\sigma^{-1})$$

Lemma 2.7.

$$\chi_V(\sigma^{-1}) = \overline{\chi_V(\sigma)}$$

Proof. The trace of a matrix A is equal to the sum of its eigenvalues λ_i. If A has finite order: $A^m = I_d$ then its eigenvalues are roots of unity. Therefore, their inverses are equal to their complex conjugates. So,

$$\mathrm{Tr}(A^{-1}) = \sum \lambda_i^{-1} = \sum \overline{\lambda_i} = \overline{\mathrm{Tr}(A)}$$

Since G is finite, the lemma follows. □

Theorem 2.8. *The character of the dual representation V^* is the complex conjugate of the character of V:*

$$\chi_{V^*}(\sigma) = \overline{\chi_V(\sigma)}$$