

I.e.,  $\sigma^*(x)$  is the coefficient of  $\sigma$  in the expansion of  $x$ . The regular character is then given by

$$\chi_{reg}(\tau) = \sum_{\sigma \in G} \sigma^*(\tau\sigma)$$

But this is clearly equal to 0 if  $\tau \neq 1$  since the coefficient of  $\sigma$  in  $\tau\sigma$  is 0. And we already know that  $\chi_{reg}(1) = \dim \mathbb{C}[G] = n$ .  $\square$

**Lemma 2.15.** *There are only finitely many isomorphism classes of simple  $G$ -modules.*

*Proof.* First choose a decomposition of the regular representation into simple modules:

$$\mathbb{C}[G] \cong \bigoplus S_\alpha$$

Then I claim that any simple module  $S$  is isomorphic to one of the  $S_\alpha$  in this decomposition. And this will prove the lemma.

To prove the claim, choose any nonzero element  $x_0 \in S$ . Then  $x_0$  generates  $S$  (the submodule generated by  $x_0$  is either 0 or  $S$ ). Therefore, we have an epimorphism

$$\phi : \mathbb{C}[G] \twoheadrightarrow S$$

given by  $\phi(r) = rx_0$ . When we restrict  $\phi$  to each simple component  $S_\alpha$ , we get a homomorphism  $\phi|_{S_\alpha} : S_\alpha \rightarrow S$  which, by Schur's lemma, must either be zero or an isomorphism. These restrictions cannot all be zero since  $\phi$  is an epimorphism. Therefore, one of them is an isomorphism  $S_\alpha \cong S$ . This proves the claim.  $\square$

This proof shows more than the lemma states. It proves:

**Lemma 2.16.** *The regular representation  $\mathbb{C}[G]$  contains an isomorphic copy of every simple  $G$ -module.*

Therefore, in order to find all the irreducible representations, we need to decompose the regular representation as a sum of simple modules.

2.2.2. *decomposition of the regular representation.* At this point I used the Wedderburn structure theorem again:

$$\mathbb{C}[G] \cong \prod_{i=1}^b \text{Mat}_{d_i}(\mathbb{C}) = \prod R_i$$

where, following Lang, we write  $R_i = \text{Mat}_{d_i}(\mathbb{C})$ .

Let  $S_i = \mathbb{C}^{d_i}$  be the vector space of column vectors. Then  $R_i$  acts on the left by matrix multiplication and it is easy to see that  $S_i$  is a simple  $R_i$ -module since it is generated by any nonzero element.

**Lemma 2.17.** *If  $\phi : R' \rightarrow R$  is an epimorphism of rings and  $S$  is any simple  $R$ -module, then  $S$  becomes a simple  $R'$ -module with the action of  $R'$  induced by  $\phi$ .*

*Proof.* If  $x_0 \in S$  is any nonzero element then  $R'x_0 = Rx_0 = S$ . So, any nonzero element of  $S$  generates the whole thing as an  $R'$ -module. So, it is simple.  $\square$

Since  $\mathbb{C}[G] = \prod R_i$ , we can make  $S_i$  into a  $G$ -module with the ring homomorphism:

$$\bar{\rho}_i : \mathbb{C}[G] \xrightarrow{\pi_i} R_i \xrightarrow{\cong} \text{End}_{\mathbb{C}}(S_i)$$

Since  $\pi_i : \mathbb{C}[G] \rightarrow R_i$  is an epimorphism,  $S_i$  becomes a simple  $G$ -module. In other words, the corresponding representation is irreducible:

$$\rho_i : G \rightarrow \text{Aut}_{\mathbb{C}}(S_i)$$

Also, Lang points out that

$$R_j S_i = 0$$

if  $i \neq j$ . (And  $R_i S_i = S_i$ .) This is the key point. It shows immediately that the  $G$ -modules  $S_i$  are not isomorphic. And it will also show that the characters are linearly independent.

In order to show that the characters

$$\chi_i = \chi_{\rho_i} = \chi_{S_i}$$

are linearly independent we will evaluate them on the central idempotents  $e_i$  corresponding to the decomposition  $\mathbb{C}[G] = \prod R_i$ . As we discussed earlier, this product decomposition gives a decomposition of unity:

$$1 = e_1 + \cdots + e_b$$

where  $e_i$  is the unity of  $R_i$ . (We want to say “ $e_i = 1$ ” but there would be too many 1’s.) We then need to compute  $\chi_i(e_j)$ . But this is not defined since  $e_j$  is not an element of  $G$ . We need to extend  $\chi_i$  to a map on  $\mathbb{C}[G]$ .

**2.2.3. linear extension of characters.** If  $\chi : G \rightarrow \mathbb{C}$  is any character, we define the *linear extension* of  $\chi$  to  $\mathbb{C}[G]$  by the formula

$$\chi \left( \sum a_{\sigma} \sigma \right) = \sum a_{\sigma} \chi(\sigma)$$

Since the symbol  $\bar{\chi}$  is already taken ( $\bar{\chi}$  is the complex conjugate of  $\chi$ ), I decided to use the same symbol  $\chi$  to denote the linear extension of  $\chi$  given by the above formula.

The linear extension of  $\chi_\rho$  is  $\chi_{\bar{\rho}}$  which is the trace of the linear extension of  $\rho$ . To see this let  $x = \sum a_\sigma \sigma$ . Then

$$\sum a_\sigma \chi_\rho(\sigma) = \sum a_\sigma \operatorname{Tr}(\rho(\sigma)) = \operatorname{Tr}\left(\sum a_\sigma \rho(\sigma)\right) = \operatorname{Tr}(\bar{\rho}(x))$$

**Lemma 2.18.**

$$\chi_i(e_j) = \begin{cases} 0 & \text{if } i \neq j \\ d_i & \text{if } i = j \end{cases}$$

*Proof.* If  $i \neq j$  we have

$$\chi_i(e_j) = \operatorname{Tr}(\rho_i(e_j)) = 0$$

since  $\rho_i(e_j)$  is the zero matrix (giving the action of  $e_j \in R_j$  on  $W_i$ ).

If  $i = j$  then

$$\chi_i(e_i) = \dim S_i = d_i$$

since  $e_i$  is unity in  $R_i$ . □

This proves the second part of Theorem 2.11: If  $\sum a_i \chi_i = 0$  then

$$\sum a_i \chi_i(e_j) = a_j d_j = 0$$

which forces  $a_j = 0$  for all  $j$ .

**Theorem 2.19.** *The regular representation decomposes as:*

$$\mathbb{C}[G] \cong \sum_{i=1}^b d_i S_i$$

*Proof.* The  $i$ th block of the Wedderburn decomposition is a  $d_i \times d_i$  matrix which, as a left module, decomposes into  $d_i$  column vectors, i.e., into a direct sum of  $d_i$  copies of the simple module  $d_i$ . □

2.2.4. *example.* Take  $G = S_3$ . Then we already saw that

$$\mathbb{C}[S_3] \cong \mathbb{C} \times \mathbb{C} \times \operatorname{Mat}_2(\mathbb{C})$$

So, there are three simple modules  $S_1, S_2, S_3$

$S_1 = \mathbb{C}$  is the trivial representation.

$S_2$  is the sign representation  $\rho_2(\sigma) = \operatorname{sgn}(\sigma) = \pm 1$

$S_3$  is a simple 2-dimensional module.

Since characters are class functions, their value is the same on conjugate elements. So, we only need their values on representatives  $1, (12), (123)$ . The characters  $\chi_1, \chi_2$  are easy to compute. The last irreducible character is determined by the equation

$$\chi_{\text{reg}} = \chi_1 + \chi_2 + 2\chi_3$$

So, here is the character table of  $S_3$ :

	1	(12)	(123)
$\chi_1$	1	1	1
$\chi_2$	1	-1	1
$\chi_3$	2	0	-1
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$\chi_{reg}$	6	0	0

All characters of  $S_3$  are nonnegative integer linear combinations of  $\chi_1, \chi_2, \chi_3$ .

**2.3. formula for idempotents.** Lang gives a formula for the idempotents  $e_i \in \mathbb{C}[G]$  in terms of the corresponding irreducible character  $\chi_i$ . The key point is that the linear extension

$$\bar{\rho}_i : \mathbb{C}[G] \rightarrow \text{End}_{\mathbb{C}}(S_i) = \text{Mat}_{d_i}(\mathbb{C})$$

of  $\rho_i$  sends  $e_i$  to the identity matrix. Therefore,

$$\bar{\rho}_i(e_i\sigma) = \bar{\rho}_i(e_i)\rho_i(\sigma) = \rho_i(\sigma)$$

Also,  $\bar{\rho}_i(e_j\sigma)$  is the zero matrix if  $i \neq j$ . Therefore,

$$\chi_i(e_j\sigma) = \begin{cases} \rho_i(\sigma) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Now use the regular character. If  $e_i = \sum a_{\tau}\tau$  then

$$a_{\tau} = \frac{1}{n}\chi_{\text{reg}}(e_i\tau^{-1}) = \frac{1}{n}\sum_j d_j\chi_j(e_i\tau^{-1}) = \frac{d_i}{n}\chi_i(\tau^{-1})$$

**Theorem 2.20.**

$$e_i = \frac{d_i}{n}\sum_{\tau \in G}\chi_i(\tau^{-1})\tau$$

This formula has an important consequence.

**Corollary 2.21.**  $d_i|n$  (Each  $d_i$  divides  $n = |G|$ .)

*Proof.* First recall that  $e_i$  is an idempotent. So,

$$e_i = e_i^2 = \frac{d_i}{n}\sum_{\tau \in G}\chi_i(\tau^{-1})\tau e_i$$

Now multiply by  $n/d_i$  to get:

$$(2.1) \quad \frac{n}{d_i}e_i = \sum_{\tau \in G}\chi_i(\tau^{-1})\tau e_i$$

I mentioned earlier that  $\chi_i(\tau^{-1}) = \sum \lambda_j$  is a sum of  $m$ th roots of unity where  $m = o(\tau^{-1}) = o(\tau)$ . But this number divides  $n = |G|$ . So, each  $\lambda_j$  is a power of  $\zeta = e^{2\pi i/n}$

Let  $M_i \subset \mathbb{C}[G]$  be the additive subgroup generated by all elements of the form  $\zeta^j\tau e_i$  (for all  $j$  and fixed  $i$ ). This is a finitely generated torsion free (and thus free)  $\mathbb{Z}$ -module and equation (2.1) shows that  $M_i$  is invariant under multiplication by the rational number  $n/d_i$ . Therefore,  $n/d_i$  is integral. Since  $\mathbb{Z}$  is integrally closed in  $\mathbb{Q}$  this implies that  $n/d_i \in \mathbb{Z}$ .  $\square$

**2.4. character tables.** I decided to construct some character tables (as I did for  $G = S_3$ ) and explain properties of characters using the examples. The *character table* is defined to be the  $b \times b$  matrix with entries  $\chi_i(c_j)$  where  $c_j$  is the  $j$ th conjugacy class. The characters are usually arranged in order of degree  $d_i$  with  $\chi_1$  being the trivial character. The conjugacy classes are arranged arbitrarily with  $c_1 = \{1\}$ . So, the character table looks like this:

	1	$c_2$	$c_3$	$\cdots$	$c_b$
$\chi_1$	1	1	1	$\cdots$	1
$\chi_2$	$d_2$				
$\chi_3$	$d_3$				
$\cdots$	$\cdots$			$\chi_i(c_j)$	
$\chi_b$	$d_b$				

**2.4.1. one-dimensional characters.** The case  $d = 1$  is very special. First of all, any one-dimensional representation of  $G$  is irreducible. So, it is one of the  $\rho_i$ . Here are all the things I pointed out:

**Proposition 2.22.** *Suppose that  $d_i = 1$ . Then*

- (1)  $\chi_i = \rho_i$ : *The character is the representation.*
- (2)  $\chi_i(\sigma)$  *is an  $m$ th root of unity where  $m = o(\sigma)$ .*
- (3)  $\chi_i(\sigma\tau) = \chi_i(\sigma)\chi_i(\tau)$ .

*Proof.* This hardly need proof. When  $d_i = 1$ , the representation is:

$$\rho_i : G \rightarrow \text{Aut}_{\mathbb{C}}(S_i) = GL_1(\mathbb{C}) = \mathbb{C}^\times$$

The trace of a  $1 \times 1$  matrix is equal to the matrix itself. So,  $\chi_i(\sigma) = \rho_i(\sigma)$ . Since  $\rho_i$  is a homomorphism, so is  $\chi_i$ . This means  $\chi_i$  is multiplicative. Also,  $\sigma^m = 1$  implies that  $\chi_i(\sigma)^m = 1$ . □

**2.4.2. example:  $\mathbb{Z}/3$ .** Since  $\mathbb{Z}/3 = \{1, \sigma, \sigma^2\}$  is an abelian group we have  $b = c = n = 3$ . Every element is its own conjugacy class. Also, all blocks have size  $d_i = 1$ . This gives the following partial character table.

	1	$\sigma$	$\sigma^2$
$\chi_1$	1	1	1
$\chi_2$	1		
$\chi_3$	1		

From our discussion of one-dimensional characters we know that each  $\chi_i(\sigma)$  is a third root of unity:

$$\chi_i(\sigma) = 1, \omega, \omega^2$$