

2.2. Irreducible characters.

Definition 2.9. A representation $\rho : G \rightarrow \text{Aut}_{\mathbb{C}}(V)$ is called irreducible if V is a simple G -module. The character

$$\chi_{\rho} = \chi_V : G \rightarrow \mathbb{C}$$

of an irreducible representation is called an irreducible character.

Theorem 2.10. Every character is a nonnegative integer linear combination of irreducible characters.

Proof. Since $\mathbb{C}[G]$ is semisimple, any G -module V is a direct sum of simple modules $V \cong \bigoplus S_{\alpha}$. So, the character of V is a sum of the corresponding irreducible characters: $\chi_V = \sum \chi_{S_{\alpha}}$. \square

If we collect together multiple copies of the same simple module we get $V = \bigoplus n_i S_i$ and

$$\chi_V = \sum_{i=1}^r n_i \chi_i$$

where χ_i is the character of S_i . This makes sense only if we know that there are only finitely many nonisomorphic simple modules S_i and that the corresponding characters χ_i are distinct functions $G \rightarrow \mathbb{C}$. In fact we will show the following.

Theorem 2.11. (1) There are exactly b (the number of blocks) irreducible representations S_i up to isomorphism.
 (2) The corresponding characters χ_i are linearly independent.

This will immediately imply the following.

Corollary 2.12. The irreducible characters χ_1, \dots, χ_b form a basis for the b -dimensional vector space of all class functions $G \rightarrow \mathbb{C}$.

2.2.1. *regular representation.* This is a particularly elementary representation and character which contains all the simple modules.

Definition 2.13. The free module ${}_{\mathbb{C}[G]}\mathbb{C}[G]$ is called the regular representation of G . The corresponding character is called the regular character: $\chi_{reg} = \chi_{\mathbb{C}[G]} : G \rightarrow \mathbb{C}$.

Theorem 2.14. $\chi_{reg}(\sigma) = \begin{cases} n = |G| & \text{if } \sigma = 1 \\ 0 & \text{if } \sigma \neq 1 \end{cases}$

Proof. I used the basis-dual basis formula for characters. The regular representation $V = \mathbb{C}[G]$ has basis elements $\sigma \in G$ and dual basis elements σ^* given by

$$\sigma^* \left(\sum a_\tau \tau \right) = a_\sigma$$

I.e., $\sigma^*(x)$ is the coefficient of σ in the expansion of x . The regular character is then given by

$$\chi_{reg}(\tau) = \sum_{\sigma \in G} \sigma^*(\tau\sigma)$$

But this is clearly equal to 0 if $\tau \neq 1$ since the coefficient of σ in $\tau\sigma$ is 0. And we already know that $\chi_{reg}(1) = \dim \mathbb{C}[G] = n$. \square

Lemma 2.15. *There are only finitely many isomorphism classes of simple G -modules.*

Proof. First choose a decomposition of the regular representation into simple modules:

$$\mathbb{C}[G] \cong \bigoplus S_\alpha$$

Then I claim that any simple module S is isomorphic to one of the S_α in this decomposition. And this will prove the lemma.

To prove the claim, choose any nonzero element $x_0 \in S$. Then x_0 generates S (the submodule generated by x_0 is either 0 or S). Therefore, we have an epimorphism

$$\phi : \mathbb{C}[G] \twoheadrightarrow S$$

given by $\phi(r) = rx_0$. When we restrict ϕ to each simple component S_α , we get a homomorphism $\phi|_{S_\alpha} : S_\alpha \rightarrow S$ which, by Schur's lemma, must either be zero or an isomorphism. These restrictions cannot all be zero since ϕ is an epimorphism. Therefore, one of them is an isomorphism $S_\alpha \cong S$. This proves the claim. \square

This proof shows more than the lemma states. It proves:

Lemma 2.16. *The regular representation $\mathbb{C}[G]$ contains an isomorphic copy of every simple G -module.*

Therefore, in order to find all the irreducible representations, we need to decompose the regular representation as a sum of simple modules.

2.2.2. *decomposition of the regular representation.* At this point I used the Wedderburn structure theorem again:

$$\mathbb{C}[G] \cong \prod_{i=1}^b \text{Mat}_{d_i}(\mathbb{C}) = \prod R_i$$

where, following Lang, we write $R_i = \text{Mat}_{d_i}(\mathbb{C})$.

Let $S_i = \mathbb{C}^{d_i}$ be the vector space of column vectors. Then R_i acts on the left by matrix multiplication and it is easy to see that S_i is a simple R_i -module since it is generated by any nonzero element.

Lemma 2.17. *If $\phi : R' \rightarrow R$ is an epimorphism of rings and S is any simple R -module, then S becomes a simple R' -module with the action of R' induced by ϕ .*

Proof. If $x_0 \in S$ is any nonzero element then $R'x_0 = Rx_0 = S$. So, any nonzero element of S generates the whole thing as an R' -module. So, it is simple. \square

Since $\mathbb{C}[G] = \prod R_i$, we can make S_i into a G -module with the ring homomorphism:

$$\bar{\rho}_i : \mathbb{C}[G] \xrightarrow{\pi_i} R_i \xrightarrow{\cong} \text{End}_{\mathbb{C}}(S_i)$$

Since $\pi_i : \mathbb{C}[G] \rightarrow R_i$ is an epimorphism, S_i becomes a simple G -module. In other words, the corresponding representation is irreducible:

$$\rho_i : G \rightarrow \text{Aut}_{\mathbb{C}}(S_i)$$

Also, Lang points out that

$$R_j S_i = 0$$

if $i \neq j$. (And $R_i S_i = S_i$.) This is the key point. It shows immediately that the G -modules S_i are not isomorphic. And it will also show that the characters are linearly independent.

In order to show that the characters

$$\chi_i = \chi_{\rho_i} = \chi_{S_i}$$

are linearly independent we will evaluate them on the central idempotents e_i corresponding to the decomposition $\mathbb{C}[G] = \prod R_i$. As we discussed earlier, this product decomposition gives a decomposition of unity:

$$1 = e_1 + \cdots + e_b$$

where e_i is the unity of R_i . (We want to say “ $e_i = 1$ ” but there would be too many 1’s.) We then need to compute $\chi_i(e_j)$. But this is not defined since e_j is not an element of G . We need to extend χ_i to a map on $\mathbb{C}[G]$.

2.2.3. *linear extension of characters.* If $\chi : G \rightarrow \mathbb{C}$ is any character, we define the *linear extension* of χ to $\mathbb{C}[G]$ by the formula

$$\chi\left(\sum a_\sigma \sigma\right) = \sum a_\sigma \chi(\sigma)$$

Since the symbol $\bar{\chi}$ is already taken ($\bar{\chi}$ is the complex conjugate of χ), I decided to use the same symbol χ to denote the linear extension of χ given by the above formula.

The linear extension of χ_ρ is $\chi_{\bar{\rho}}$ which is the trace of the linear extension of ρ . To see this let $x = \sum a_\sigma \sigma$. Then

$$\sum a_\sigma \chi_\rho(\sigma) = \sum a_\sigma \operatorname{Tr}(\rho(\sigma)) = \operatorname{Tr}\left(\sum a_\sigma \rho(\sigma)\right) = \operatorname{Tr}(\bar{\rho}(x))$$

Lemma 2.18.

$$\chi_i(e_j) = \begin{cases} 0 & \text{if } i \neq j \\ d_i & \text{if } i = j \end{cases}$$

Proof. If $i \neq j$ we have

$$\chi_i(e_j) = \operatorname{Tr}(\rho_i(e_j)) = 0$$

since $\rho_i(e_j)$ is the zero matrix (giving the action of $e_j \in R_j$ on W_i).

If $i = j$ then

$$\chi_i(e_i) = \dim S_i = d_i$$

since e_i is unity in R_i . □

This proves the second part of Theorem 2.11: If $\sum a_i \chi_i = 0$ then

$$\sum a_i \chi_i(e_j) = a_j d_j = 0$$

which forces $a_j = 0$ for all j .

Theorem 2.19. *The regular representation decomposes as:*

$$\mathbb{C}[G] \cong \sum_{i=1}^b d_i S_i$$

Proof. The i th block of the Wedderburn decomposition is a $d_i \times d_i$ matrix which, as a left module, decomposes into d_i column vectors, i.e., into a direct sum of d_i copies of the simple module d_i . □

2.2.4. *example.* Take $G = S_3$. Then we already saw that

$$\mathbb{C}[S_3] \cong \mathbb{C} \times \mathbb{C} \times \text{Mat}_2(\mathbb{C})$$

So, there are three simple modules S_1, S_2, S_3

$S_1 = \mathbb{C}$ is the trivial representation.

S_2 is the sign representation $\rho_2(\sigma) = \text{sgn}(\sigma) = \pm 1$

S_3 is a simple 2-dimensional module.

Since characters are class functions, their value is the same on conjugate elements. So, we only need their values on representatives $1, (12), (123)$. The characters χ_1, χ_2 are easy to compute. The last irreducible character is determined by the equation

$$\chi_{reg} = \chi_1 + \chi_2 + 2\chi_3$$

So, here is the character table of S_3 :

	1	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1
χ_{reg}	6	0	0

All characters of S_3 are nonnegative integer linear combinations of χ_1, χ_2, χ_3 .

2.3. formula for idempotents. Lang gives a formula for the idempotents $e_i \in \mathbb{C}[G]$ in terms of the corresponding irreducible character χ_i . The key point is that the linear extension

$$\bar{\rho}_i : \mathbb{C}[G] \rightarrow \text{End}_{\mathbb{C}}(S_i) = \text{Mat}_{d_i}(\mathbb{C})$$

of ρ_i sends e_i to the identity matrix. Therefore,

$$\bar{\rho}_i(e_i\sigma) = \bar{\rho}_i(e_i)\rho_i(\sigma) = \rho_i(\sigma)$$

Also, $\bar{\rho}_i(e_j\sigma)$ is the zero matrix if $i \neq j$. Therefore,

$$\chi_i(e_j\sigma) = \begin{cases} \rho_i(\sigma) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Now use the regular character. If $e_i = \sum a_{\tau}\tau$ then

$$a_{\tau} = \frac{1}{n}\chi_{\text{reg}}(e_i\tau^{-1}) = \frac{1}{n}\sum_j d_j\chi_j(e_i\tau^{-1}) = \frac{d_i}{n}\chi_i(\tau^{-1})$$

Theorem 2.20.

$$e_i = \frac{d_i}{n}\sum_{\tau \in G} \chi_i(\tau^{-1})\tau$$

This formula has an important consequence.

Corollary 2.21. $d_i | n$ (Each d_i divides $n = |G|$.)

Proof. First recall that e_i is an idempotent. So,

$$e_i = e_i^2 = \frac{d_i}{n}\sum_{\tau \in G} \chi_i(\tau^{-1})\tau e_i$$

Now multiply by n/d_i to get:

$$(2.1) \quad \frac{n}{d_i}e_i = \sum_{\tau \in G} \chi_i(\tau^{-1})\tau e_i$$

I mentioned earlier that $\chi_i(\tau^{-1}) = \sum \lambda_j$ is a sum of m th roots of unity where $m = o(\tau^{-1}) = o(\tau)$. But this number divides $n = |G|$. So, each λ_j is a power of $\zeta = e^{2\pi i/n}$

Let $M_i \subset \mathbb{C}[G]$ be the additive subgroup generated by all elements of the form $\zeta^j\tau e_i$ (for all j and fixed i). This is a finitely generated torsion free (and thus free) \mathbb{Z} -module and equation (2.1) shows that M_i is invariant under multiplication by the rational number n/d_i . Therefore, n/d_i is integral. Since \mathbb{Z} is integrally closed in \mathbb{Q} this implies that $n/d_i \in \mathbb{Z}$. \square