

$\chi_i(\sigma^2) = \chi_i(\sigma)^2 = 1, \omega^2, \omega$ respectively. So, the complete character table is:

	1	σ	σ^2
χ_1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω

2.4.3. *example: $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.* Let's call the elements of the group $1, \sigma, \tau, \sigma\tau$. Since $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ is abelian, all characters are again one dimensional and the values must be square roots of 1, i.e., they must be ± 1 . So, we got the following.

	1	σ	τ	$\sigma\tau$
χ_1	1	1	1	1
χ_2	1	-1	1	-1
χ_3	1	-1	-1	1
χ_4	1	1	-1	-1

Each row is clearly a one-dimensional representation. There are no others because we know that there are exactly $b = 4$ such representations. So, this is the complete character table.

2.4.4. *example: D_4 .* This is the *dihedral group* of order 8 with presentation:

$$D_4 = \langle \sigma, \tau \mid \sigma^4, \tau^2, \sigma\tau\sigma\tau \rangle$$

(Replace 4 by any n to get the dihedral group of order $2n$.) To find the numbers d_i we have to write $n = 8$ as a sum of squares which are not all 1 (because D_4 is nonabelian) and so that there is at least one 1 (since $d_1 = 1$). The solution is:

$$8 = 1 + 1 + 1 + 1 + 4$$

Therefore, $b = c = 5$.

The elements of the group are:

$$D_4 = \{e, \sigma, \sigma^2, \sigma^3, \tau, \sigma\tau, \sigma^2\tau, \sigma^3\tau\}$$

Among these, σ, σ^3 are conjugate since $\tau\sigma\tau^{-1} = \sigma^3$, $\tau, \sigma^2\tau = \sigma\tau\sigma^{-1}$ are conjugate and $\sigma\tau, \sigma^3\tau = \sigma(\sigma\tau)\sigma^{-1}$ are conjugate. There are no other conjugacy relations since we got it down to 5 classes.

Among the 5 characters, the first 4 are 1-dimensional. And we can find them very quickly as follows. The center of D_4 is the set of elements which are alone in their conjugacy class. So,

$$Z(D_4) = \{1, \sigma^2\}$$

This is a normal subgroup of D_4 with quotient isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.

We already have four irreducible representations ρ_1, \dots, ρ_4 of $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. We can compose with the projection to get four irreducible representations of D_4

$$D_4 \twoheadrightarrow D_4/Z \xrightarrow{\rho_i = \chi_i} \mathbb{C}^\times$$

This gives the first four lines in the character table:

	1	σ^2	σ	τ	$\sigma\tau$
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	-1	-1	1
χ_4	1	1	1	-1	-1
χ_5	2	-2	0	0	0

To get the last line we use the equation:

$$\chi_{reg} = \sum d_i \chi_i = \chi_1 + \chi_2 + \chi_3 + \chi_4 + 2\chi_5$$

2.4.5. *kernel of a representation.* Looking at the character table, we can determine which elements of the group lie in the kernel of each representation.

Lemma 2.23. $\sigma \in \ker \rho \iff \chi_\rho(\sigma) = d = \chi_\rho(1)$.

Proof. In a d -dimensional representation, $\chi(\sigma) = \lambda_1 + \dots + \lambda_d$ is a sum of d roots of unity. This sum is equal to d if and only if every $\lambda_i = 1$ which is equivalent to saying that $\rho(\sigma)$ is the identity matrix (since $\rho(\sigma)$ has finite order). \square

Using the same argument it follows that:

Proposition 2.24. $|\chi_\rho(\sigma)| = d$ if and only if $\rho(\sigma) = \lambda I_d$ is a scalar multiple of the identity matrix. Furthermore, $\lambda = \chi_\rho(\sigma)/d$.

For example, in the last irreducible representation of D_4 we have

$$|\chi_5(\sigma^2)| = 2 = d_5$$

Therefore, $\rho_5(\sigma^2) = -I_2$.

2.4.6. *finding all normal subgroups.* Finally, I claimed that the character table determines all normal subgroups of the group G . This is based on the trick that we used to construct the character table of D_4 .

Suppose that N is a normal subgroup of G and $\rho_i, i = 1, \dots, r$ are the irreducible representations of G/N .

Lemma 2.25.

$$N = \bigcap \ker(\rho_i \circ \pi)$$

where $\pi : G \rightarrow G/N$ is the quotient map.

Proof. Let $K = \bigcap \ker(\rho_i \circ \pi)$. Then clearly, $N \subseteq K$. So, suppose that K is bigger than N . Then the representations ρ_i would all factor through the quotient G/K :

$$\rho_i : G/N \xrightarrow{\phi} G/K \xrightarrow{\psi_i} \text{Aut}_{\mathbb{C}}(S_i)$$

This is not possible because the sum of the squares of the dimensions of these representations add up to the order of G/N :

$$|G/K| < |G/N| = \sum d_i^2$$

So, the ψ_i are distinct irreducible representations of G/K whose dimensions squared add up to more than the order of the group. This contradiction proves the lemma. \square

Combining Lemmas 2.25 and 2.23, we get the following.

Theorem 2.26. *The normal subgroups of a finite group G can be determined from its character table as follows.*

- (1) *The kernel of ρ_i is the union of all conjugacy classes c_j for which $\chi_i(c_j) = d_i = \chi_i(1)$.*
- (2) *A collection of conjugacy classes forms a normal subgroup if and only if it is an intersection of kernels of irreducible representations ρ_i .*

2.5. orthogonality relations. The character table satisfies two orthogonality relations:

- (1) row orthogonality
- (2) column orthogonality

First, I will do row orthogonality. The rows are the characters χ_i . We want to show that they are “orthogonal” in some sense.

2.5.1. *main theorem and consequences.*

Definition 2.27. If $f, g : G \rightarrow \mathbb{C}$ are class functions then we define $\langle f, g \rangle \in \mathbb{C}$ by

$$\langle f, g \rangle = \langle g, f \rangle = \frac{1}{n} \sum_{\sigma \in G} f(\sigma)g(\sigma^{-1})$$

The main theorem is the following.

Theorem 2.28. *If V, W are G -modules then*

$$\langle \chi_V, \chi_W \rangle = \dim_{\mathbb{C}} \text{Hom}_G(V, W)$$

Before I prove this let me explain the consequences.

Corollary 2.29. *The rows of the character table are orthonormal in the sense that:*

$$\langle \chi_i, \chi_j \rangle = \delta_{ij}$$

Proof. It follows from Schur’s lemma that

$$\langle \chi_i, \chi_j \rangle = \dim_{\mathbb{C}} \text{Hom}_G(S_i, S_j) = \delta_{ij}$$

since $\text{Hom}_G(S_i, S_j) = 0$ for $i \neq j$ and $\text{Hom}_{\mathbb{C}}(S_i, S_i) = \mathbb{C}$. □

Since only conjugacy classes appear in the character table we have:

$$\langle \chi_i, \chi_j \rangle = \sum_{k=1}^b |c_k| \chi_i(c_k) \overline{\chi_j(c_k)}$$

For example, for $G = S_3$ we have the character table:

$ c_j $	1	3	2
	1	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

$$\langle \chi_1, \chi_2 \rangle = \frac{(1)(1) + 3(1)(-1) + 2(1)(1)}{6} = \frac{1 - 3 + 2}{6} = 0$$

This formula also tells us that a representation is determined by its character in the following way.