

Corollary 2.30. *Suppose that the semisimple decomposition of the G -module V is $V = \sum n_i S_i$. Then*

$$n_i = \langle \chi_V, \chi_i \rangle$$

Proof. Since $\chi_{V \oplus W} = \chi_V + \chi_W$, we have: $\chi_V = \sum n_j \chi_j$. So,

$$\langle \chi_V, \chi_i \rangle = \left\langle \sum n_j \chi_j, \chi_i \right\rangle = n_i$$

□

2.5.2. *proof of the main theorem.* The theorem will follow from three lemmas. The first lemma calculates the dimension of the fixed point set of V .

Definition 2.31. If V is a G -module then the *fixed point set* of the action of G is given by

$$V^G := \{v \in V \mid \sigma v = v \ \forall \sigma \in G\}$$

Lemma 2.32. *The dimension of the fixed point set is equal to the average value of the corresponding character:*

$$\dim_{\mathbb{C}} V^G = \frac{1}{n} \sum_{\sigma \in G} \chi_V(\sigma)$$

Proof. The projection map

$$\pi : V \rightarrow V^G$$

is given by

$$\pi(v) = \frac{1}{n} \sum \sigma v$$

It is clear that

- (1) $\pi(v) \in V^G$ since multiplication by any $\tau \in G$ will just permute the summands.
- (2) $\pi(v) = v$ if $v \in V^G$ because, in that case, each $\sigma v = v$ and there are n terms.

Therefore, π is a projection map, i.e., a linear retraction onto V^G . Looking at the formula we see that π is multiplication by the idempotent $e_1 = \frac{1}{n} \sum_{\sigma \in G} \sigma$. (This is the idempotent corresponding to the trivial representation.) So:

$$\dim V^G = \text{Tr}(\pi) = \chi_V(e_1) = \chi_V \left(\frac{1}{n} \sum_{\sigma \in G} \sigma \right) = \frac{1}{n} \sum_{\sigma \in G} \chi_V(\sigma)$$

Explanations:

(1) $\dim V^G = \text{Tr}(\pi)$ because $V \cong V^G \oplus W$ ($W = \ker \pi$). So, the matrix of π is:

$$\pi = \begin{pmatrix} 1_{V^G} & 0 \\ 0 & 0_W \end{pmatrix}$$

making $\text{Tr}(\pi) = \text{Tr}(1_{V^G}) = \dim_{\mathbb{C}} V^G$.

(2) $\text{Tr}(\pi) = \chi_V(e_1)$ by definition of the character:

$$\chi_V(e_1) := \text{Tr}(e_1 \cdot : V \rightarrow V)$$

This is the trace of the mapping $V \rightarrow V$ given by multiplication by e_1 . But we are calling that mapping π . \square

Lemma 2.33. *If V, W are representations of G then*

$$\text{Hom}_G(V, W) = \text{Hom}_{\mathbb{C}}(V, W)^G$$

where G acts on $\text{Hom}_{\mathbb{C}}(V, W)$ by conjugation, i.e., $\sigma f = \sigma \circ f \circ \sigma^{-1}$ which means that

$$(\sigma f)(v) = \sigma f(\sigma^{-1}v)$$

Proof. This is trivial. Given any linear map $f : V \rightarrow W$, f is a G -homomorphism iff

$$\sigma \circ f = f \circ \sigma \iff \sigma \circ f \circ \sigma^{-1} = f \iff \sigma f = f$$

iff $f \in \text{Hom}_{\mathbb{C}}(V, W)^G$. \square

Lemma 2.34. $\text{Hom}_{\mathbb{C}}(V, W) \cong V^* \otimes W$ as G -modules.

Proof. Let $\phi : V^* \otimes W \rightarrow \text{Hom}_{\mathbb{C}}(V, W)$ be given by

$$\phi(f \otimes w)(v) = f(v)w$$

To check that this is a G -homomorphism we need to show that $\phi\sigma = \sigma\phi$ for any $\sigma \in G$. So, we compute both sides:

$$\phi\sigma(f \otimes w) = \phi(\sigma f \otimes \sigma w) = \phi(f \circ \sigma^{-1} \otimes \sigma w)$$

which sends $v \in V$ to

$$\phi(f \circ \sigma^{-1} \otimes \sigma w)(v) = f(\sigma^{-1}v)\sigma w$$

On the other side we have:

$$\sigma\phi(f \otimes w) = \sigma \circ \phi(f \otimes w) \circ \sigma^{-1}$$

which also sends $v \in V$ to

$$\sigma \circ \phi(f \otimes w) \circ \sigma^{-1}v = \sigma(f(\sigma^{-1}v)w) = f(\sigma^{-1}v)\sigma w$$

This shows that ϕ commutes with the action of G . The fact that ϕ is an isomorphism is well-known: If v_i, v_i^* form a basis-dual basis pair for V and w_j form a basis for W then $v_j^* \otimes w_i$ form a basis for $V^* \otimes W$ and

$$\phi(v_j^* \otimes w_i) : v = \sum a_j v_j \mapsto v_j^*(v)w_i = a_j w_i$$

is the mapping whose matrix has ij -entry equal to 1 and all other entries 0. So, these homomorphisms form a basis for $\text{Hom}_{\mathbb{C}}(V, W)$ and ϕ is an isomorphism. \square

Proof of main theorem 2.28. Using the three lemmas we get:

$$\begin{aligned} \dim_{\mathbb{C}} \text{Hom}_G(V, W) &=_{2.33} \dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}}(V, W)^G \\ &=_{2.34} \dim_{\mathbb{C}}(V^* \otimes W)^G \\ &=_{2.32} \frac{1}{n} \sum_{\sigma \in G} \chi_{V^* \otimes W}(\sigma) \\ &= \frac{1}{n} \sum_{\sigma} \chi_{V^*}(\sigma) \chi_W(\sigma) \\ &= \frac{1}{n} \sum_{\sigma} \chi_V(\sigma^{-1}) \chi_W(\sigma) = \langle \chi_V, \chi_W \rangle \end{aligned}$$

\square

2.5.3. *character table of S_4 .* Using these formulas we can calculate the character table for S_4 . First note that there are five conjugacy classes represented by

$$1, (12), (123), (12)(34), (1234)$$

The elements of cycle form $(12)(34)$ form (with 1) a normal subgroup

$$K = \{1, (12)(34), (13)(24), (14)(23)\} \triangleleft S_4$$

called the *Klein 4-group*. The quotient S_4/K is isomorphic to the symmetric group on 3 letters. Imitating the case of D_4 , this allows us to construct the following portion of the character table for S_4 :

$ c_j $	1	6	8	3	6
	1	(12)	(123)	(12)(34)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	-1	2	0
χ_4	3				
χ_5	3				

Explanations:

- (1) Since $(12)(34) \in K$, the value of the first three characters on this conjugacy class is d_i , the same as in the first column.
- (2) Since $(1234)K = (12)K$, these two columns have the same values of χ_1, χ_2, χ_3 .

(3) Finally, the two unknown characters χ_4, χ_5 must be 3-dimensional since

$$24 = \sum d_i^2 = 1 + 1 + 4 + d_4^2 + d_5^2$$

has only one solution: $d_4 = d_5 = 3$.

To figure out the unknown characters we need another representation. The *permutation representation* P is the 4-dimensional representation of S_4 in which the elements of S_4 act by permuting the unit coordinate vectors. For example

$$\rho_P(12) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note that the trace of $\rho_P(\sigma)$ is equal to the number of letters left fixed by σ . So, χ_P takes values 4, 2, 1, 0, 0 as shown:

$ c_j $	1	6	8	3	6
	1	(12)	(123)	(12)(34)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	-1	2	0
χ_P	4	2	1	0	0
$\chi_V = \chi_P - \chi_1$	3	1	0	-1	-1

The representation P contains one copy of the trivial representation and no copies of the other two:

$$\langle \chi_P, \chi_1 \rangle = \frac{1}{24}(4 + 6(2) + 8(1)) = 1$$

$$\langle \chi_P, \chi_2 \rangle = \frac{1}{24}(4 + 6(-1)(2) + 8(1)(1)) = 0$$

$$\langle \chi_P, \chi_3 \rangle = \frac{1}{24}((2)(4) + 8(-1)(1)) = 0$$

So, $P \cong S_1 \oplus V$ where V is a 3-dimensional module which does not contain S_1, S_2 or S_3 . So, $V = nS_4 \oplus mS_5$. But S_4, S_5 are both 3-dimensional. So, $V = S_4$ (or S_5).

Using the fact that

$$\chi_1 + \chi_2 + 2\chi_3 + 3\chi_4 + 3\chi_5 = \chi_{reg}$$

we can now complete the character table of S_4 :

$ c_j $	1	6	8	3	6
	1	(12)	(123)	(12)(34)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	-1	2	0
χ_4	3	1	0	-1	-1
χ_5	3	-1	0	-1	1

From the character table of S_4 we can find all normal subgroups. First, the kernels of the 5 irreducible representations are:

- (1) $\ker \rho_1 = S_4$.
- (2) $\ker \rho_2 = A_4$ containing the conjugacy classes of 1, (123), (12)(34).
- (3) $\ker \rho_3 = K$ containing 1, (12)(34) and conjugates.
- (4) $\ker \rho_4 = 1$. I.e., ρ_4 is a faithful representation.
- (5) $\ker \rho_5 = 1$. So, ρ_5 is also faithful.

Since these subgroups contain each other:

$$1 < K < A_4 < S_4$$

intersecting them will not give any other subgroups. So, these are the only normal subgroups of S_4 .

2.5.4. *column orthogonality.* The columns of the character table also satisfy an orthogonality condition. To see it we first have to write the row orthogonality condition

$$\langle \chi_i, \chi_j \rangle = \sum_{k=1}^b \frac{|c_k|}{n} \chi_i(c_k) \overline{\chi_j(c_k)} = \delta_{ij}$$

and write it in matrix form:

$$T \begin{pmatrix} \frac{|c_1|}{n} & & 0 \\ & \ddots & \\ 0 & & \frac{|c_b|}{n} \end{pmatrix} \overline{T}^t = I_b$$

where T is the character table $T = (\chi_i(c_j))$. This equation shows that the character table T is an invertible matrix with inverse

$$T^{-1} = D \overline{T}^t$$

where D is the diagonal matrix with diagonal entries $\frac{|c_i|}{n}$. Multiplying both sides of this equation on the right by T and on the left with D^{-1} and we get:

$$\overline{T}^t T = D^{-1} = \begin{pmatrix} \frac{n}{|c_1|} & & 0 \\ & \ddots & \\ 0 & & \frac{n}{|c_b|} \end{pmatrix}$$

Looking at the entries of these matrices we get the column orthogonality relation:

Theorem 2.35. *If $\sigma, \tau \in G$ then*

$$\sum_{i=1}^b \overline{\chi_i(\sigma)} \chi_i(\tau) = \begin{cases} \frac{n}{|c|} & \text{if } \sigma, \tau \text{ are conjugate} \\ 0 & \text{if not} \end{cases}$$

Here $|c|$ is the number of conjugates of σ in G . (So, $n/|c|$ is the order of the centralizer $C(\sigma) = \{\tau \in G \mid \sigma\tau = \tau\sigma\}$ of σ .)

Corollary 2.36. *The character table $T = (\chi_i(c_j))$ determines the size of each conjugacy class c_j .*

Proof. Taking $\sigma = \tau$ in the above theorem we get

$$|C(\sigma)| = \sum_i \|\chi_i(\sigma)\|^2$$

The size of the conjugacy class c of σ is the index of its centralizer: $|c| = |G : C(\sigma)| = n/|C(\sigma)|$. \square

As an example, look at the character table for S_3 :

	1	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Column orthogonality means that the usual Hermitian dot product of the columns is zero. For example, the dot product of the first and third column is

$$(1)(1) + (1)(1) + (2)(-1) = 0$$

Also the dot product of the j th vector with itself (its length squared) is equal to $n/|c_j|$. For example, the length squared of the third column vector is

$$1 + 1 + 1 = 3$$

Making the number of conjugates of (123) equal to $6/3 = 2$.

3. INDUCTION

If H is a subgroup of G then any representation of G will restrict to a representation of H by composition:

$$H \hookrightarrow G \xrightarrow{\rho} \text{Aut}_{\mathbb{C}}(V)$$

Induction is a more complicated process which goes the other way: It starts with a representation of H and produces a representation of G . Following Lang, I will construct the same object in several different ways starting with an elementary equation for the induced character.

3.1. induced characters.

Definition 3.1. Suppose that $H \leq G$ (H is a subgroup of G) and $\chi : H \rightarrow \mathbb{C}$ is a character (or any class function). Then the *induced character*

$$\text{Ind}_H^G \chi : G \rightarrow \mathbb{C}$$

is the class function on G defined by

$$\text{Ind}_H^G \chi(\sigma) = \frac{1}{|H|} \sum_{\tau \in G} \chi(\tau\sigma\tau^{-1})$$

where $\chi(\sigma) = 0$ if $\sigma \notin H$.

The main theorem about the induced character is the following.

Theorem 3.2. *If V is any representation of H then there exists a representation W of G so that*

$$\chi_W = \text{Ind}_H^G \chi_V$$

Furthermore, W is unique up to isomorphism.

The representation W is written $W = \text{Ind}_H^G V$ and is called the *induced representation*. We will study that tomorrow.

Before proving this theorem let me give two examples.

3.1.1. *example 1.* Here is a trivial observation.

Proposition 3.3. *If G is abelian then*

$$\text{Ind}_H^G \chi(\sigma) = |G : H| \chi(\sigma)$$

Now suppose that $G = \mathbb{Z}/4 = \{1, \sigma, \sigma^2, \sigma^3\}$ and $H = \{1, \tau\}$ with $\tau = \sigma^2$. Then the character table of $H \cong \mathbb{Z}/2$ is

$H = \mathbb{Z}/2$	1	τ
χ_+	1	1
χ_-	1	-1