

**2.5. orthogonality relations.** The character table satisfies two orthogonality relations:

- (1) row orthogonality
- (2) column orthogonality

First, I will do row orthogonality. The rows are the characters  $\chi_i$ . We want to show that they are “orthogonal” in some sense.

2.5.1. *main theorem and consequences.*

**Definition 2.27.** If  $f, g : G \rightarrow \mathbb{C}$  are class functions then we define  $\langle f, g \rangle \in \mathbb{C}$  by

$$\langle f, g \rangle = \langle g, f \rangle = \frac{1}{n} \sum_{\sigma \in G} f(\sigma)g(\sigma^{-1})$$

The main theorem is the following.

**Theorem 2.28.** If  $V, W$  are  $G$ -modules then

$$\langle \chi_V, \chi_W \rangle = \dim_{\mathbb{C}} \text{Hom}_G(V, W)$$

Before I prove this let me explain the consequences.

**Corollary 2.29.** The rows of the character table are orthonormal in the sense that:

$$\langle \chi_i, \chi_j \rangle = \delta_{ij}$$

*Proof.* It follows from Schur’s lemma that

$$\langle \chi_i, \chi_j \rangle = \dim_{\mathbb{C}} \text{Hom}_G(S_i, S_j) = \delta_{ij}$$

since  $\text{Hom}_G(S_i, S_j) = 0$  for  $i \neq j$  and  $\text{Hom}_{\mathbb{C}}(S_i, S_i) = \mathbb{C}$ . □

Since only conjugacy classes appear in the character table we have:

$$\langle \chi_i, \chi_j \rangle = \sum_{k=1}^b |c_k| \chi_i(c_k) \overline{\chi_j(c_k)}$$

For example, for  $G = S_3$  we have the character table:

$ c_j $	1	3	2
	1	(12)	(123)
$\chi_1$	1	1	1
$\chi_2$	1	-1	1
$\chi_3$	2	0	-1

$$\langle \chi_1, \chi_2 \rangle = \frac{(1)(1) + 3(1)(-1) + 2(1)(1)}{6} = \frac{1 - 3 + 2}{6} = 0$$

This formula also tells us that a representation is determined by its character in the following way.

**Corollary 2.30.** *Suppose that the semisimple decomposition of the  $G$ -module  $V$  is  $V = \sum n_i S_i$ . Then*

$$n_i = \langle \chi_V, \chi_i \rangle$$

*Proof.* Since  $\chi_{V \oplus W} = \chi_V + \chi_W$ , we have:  $\chi_V = \sum n_j \chi_j$ . So,

$$\langle \chi_V, \chi_i \rangle = \left\langle \sum n_j \chi_j, \chi_i \right\rangle = n_i$$

□

2.5.2. *proof of the main theorem.* The theorem will follow from three lemmas. The first lemma calculates the dimension of the fixed point set of  $V$ .

**Definition 2.31.** *If  $V$  is a  $G$ -module then the fixed point set of the action of  $G$  is given by*

$$V^G := \{v \in V \mid \sigma v = v \ \forall \sigma \in G\}$$

**Lemma 2.32.** *The dimension of the fixed point set is equal to the average value of the corresponding character:*

$$\dim_{\mathbb{C}} V^G = \frac{1}{n} \sum_{\sigma \in G} \chi_V(\sigma)$$

*Proof.* The projection map

$$\pi : V \rightarrow V^G$$

is given by

$$\pi(v) = \frac{1}{n} \sum \sigma v$$

It is clear that

- (1)  $\pi(v) \in V^G$  since multiplication by any  $\tau \in G$  will just permute the summands.
- (2)  $\pi(v) = v$  if  $v \in V^G$  because, in that case, each  $\sigma v = v$  and there are  $n$  terms.

Therefore,  $\pi$  is a projection map, i.e., a linear retraction onto  $V^G$ . Looking at the formula we see that  $\pi$  is multiplication by the idempotent  $e_1 = \frac{1}{n} \sum_{\sigma \in G} \sigma$ . (This is the idempotent corresponding to the trivial representation.) So:

$$\dim V^G = \text{Tr}(\pi) = \chi_V(e_1) = \chi_V \left( \frac{1}{n} \sum_{\sigma \in G} \sigma \right) = \frac{1}{n} \sum_{\sigma \in G} \chi_V(\sigma)$$

Explanations:

(1)  $\dim V^G = \text{Tr}(\pi)$  because  $V \cong V^G \oplus W$  ( $W = \ker \pi$ ). So, the matrix of  $\pi$  is:

$$\pi = \begin{pmatrix} 1_{V^G} & 0 \\ 0 & 0_W \end{pmatrix}$$

making  $\text{Tr}(\pi) = \text{Tr}(1_{V^G}) = \dim_{\mathbb{C}} V^G$ .

(2)  $\text{Tr}(\pi) = \chi_V(e_1)$  by definition of the character:

$$\chi_V(e_1) := \text{Tr}(e_1 \cdot : V \rightarrow V)$$

This is the trace of the mapping  $V \rightarrow V$  given by multiplication by  $e_1$ . But we are calling that mapping  $\pi$ .  $\square$

**Lemma 2.33.** *If  $V, W$  are representations of  $G$  then*

$$\text{Hom}_G(V, W) = \text{Hom}_{\mathbb{C}}(V, W)^G$$

where  $G$  acts on  $\text{Hom}_{\mathbb{C}}(V, W)$  by conjugation, i.e.,  $\sigma f = \sigma \circ f \circ \sigma^{-1}$  which means that

$$(\sigma f)(v) = \sigma f(\sigma^{-1}v)$$

*Proof.* This is trivial. Given any linear map  $f : V \rightarrow W$ ,  $f$  is a  $G$ -homomorphism iff

$$\sigma \circ f = f \circ \sigma \iff \sigma \circ f \circ \sigma^{-1} = f \iff \sigma f = f$$

iff  $f \in \text{Hom}_{\mathbb{C}}(V, W)^G$ .  $\square$

**Lemma 2.34.**  *$\text{Hom}_{\mathbb{C}}(V, W) \cong V^* \otimes W$  as  $G$ -modules.*

*Proof.* Let  $\phi : V^* \otimes W \rightarrow \text{Hom}_{\mathbb{C}}(V, W)$  be given by

$$\phi(f \otimes w)(v) = f(v)w$$

To check that this is a  $G$ -homomorphism we need to show that  $\phi\sigma = \sigma\phi$  for any  $\sigma \in G$ . So, we compute both sides:

$$\phi\sigma(f \otimes w) = \phi(\sigma f \otimes \sigma w) = \phi(f \circ \sigma^{-1} \otimes \sigma w)$$

which sends  $v \in V$  to

$$\phi(f \circ \sigma^{-1} \otimes \sigma w)(v) = f(\sigma^{-1}v)\sigma w$$

On the other side we have:

$$\sigma\phi(f \otimes w) = \sigma \circ \phi(f \otimes w) \circ \sigma^{-1}$$

which also sends  $v \in V$  to

$$\sigma \circ \phi(f \otimes w) \circ \sigma^{-1}v = \sigma(f(\sigma^{-1}v)w) = f(\sigma^{-1}v)\sigma w$$

This shows that  $\phi$  commutes with the action of  $G$ . The fact that  $\phi$  is an isomorphism is well-known: If  $v_i, v_i^*$  form a basis-dual basis pair for  $V$  and  $w_j$  form a basis for  $W$  then  $v_j^* \otimes w_i$  form a basis for  $V^* \otimes W$  and

$$\phi(v_j^* \otimes w_i) : v = \sum a_j v_j \mapsto v_j^*(v)w_i = a_j w_i$$

is the mapping whose matrix has  $ij$ -entry equal to 1 and all other entries 0. So, these homomorphisms form a basis for  $\text{Hom}_{\mathbb{C}}(V, W)$  and  $\phi$  is an isomorphism.  $\square$

*Proof of main theorem 2.28.* Using the three lemmas we get:

$$\begin{aligned} \dim_{\mathbb{C}} \text{Hom}_G(V, W) &=_{2.33} \dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}}(V, W)^G \\ &=_{2.34} \dim_{\mathbb{C}}(V^* \otimes W)^G \\ &=_{2.32} \frac{1}{n} \sum_{\sigma \in G} \chi_{V^* \otimes W}(\sigma) \\ &= \frac{1}{n} \sum_{\sigma} \chi_{V^*}(\sigma) \chi_W(\sigma) \\ &= \frac{1}{n} \sum_{\sigma} \chi_V(\sigma^{-1}) \chi_W(\sigma) = \langle \chi_V, \chi_W \rangle \end{aligned}$$

$\square$

2.5.3. *character table of  $S_4$ .* Using these formulas we can calculate the character table for  $S_4$ . First note that there are five conjugacy classes represented by

$$1, (12), (123), (12)(34), (1234)$$

The elements of cycle form  $(12)(34)$  form (with 1) a normal subgroup

$$K = \{1, (12)(34), (13)(24), (14)(23)\} \triangleleft S_4$$

called the *Klein 4-group*. The quotient  $S_4/K$  is isomorphic to the symmetric group on 3 letters. Imitating the case of  $D_4$ , this allows us

to construct the following portion of the character table for  $S_4$ :

$ c_j $	1	6	8	3	6
	1	(12)	(123)	(12)(34)	(1234)
$\chi_1$	1	1	1	1	1
$\chi_2$	1	-1	1	1	-1
$\chi_3$	2	0	-1	2	0
$\chi_4$	3				
$\chi_5$	3				

Explanations:

- (1) Since  $(12)(34) \in K$ , the value of the first three characters on this conjugacy class is  $d_i$ , the same as in the first column.
- (2) Since  $(1234)K = (12)K$ , these two columns have the same values of  $\chi_1, \chi_2, \chi_3$ .
- (3) Finally, the two unknown characters  $\chi_4, \chi_5$  must be 3-dimensional since

$$24 = \sum d_i^2 = 1 + 1 + 4 + d_4^2 + d_5^2$$

has only one solution:  $d_4 = d_5 = 3$ .

To figure out the unknown characters we need another representation. The *permutation representation*  $P$  is the 4-dimensional representation of  $S_4$  in which the elements of  $S_4$  act by permuting the unit coordinate vectors. For example

$$\rho_P(12) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note that the trace of  $\rho_P(\sigma)$  is equal to the number of letters left fixed by  $\sigma$ . So,  $\chi_P$  takes values 4, 2, 1, 0, 0 as shown:

$ c_j $	1	6	8	3	6
	1	(12)	(123)	(12)(34)	(1234)
$\chi_1$	1	1	1	1	1
$\chi_2$	1	-1	1	1	-1
$\chi_3$	2	0	-1	2	0
$\chi_P$	4	2	1	0	0
$\chi_V = \chi_P - \chi_1$	3	1	0	-1	-1

The representation  $P$  contains one copy of the trivial representation and no copies of the other two:

$$\begin{aligned}\langle \chi_P, \chi_1 \rangle &= \frac{1}{24}(4 + 6(2) + 8(1)) = 1 \\ \langle \chi_P, \chi_2 \rangle &= \frac{1}{24}(4 + 6(-1)(2) + 8(1)(1)) = 0 \\ \langle \chi_P, \chi_3 \rangle &= \frac{1}{24}((2)(4) + 8(-1)(1)) = 0\end{aligned}$$

So,  $P \cong S_1 \oplus V$  where  $V$  is a 3-dimensional module which does not contain  $S_1, S_2$  or  $S_3$ . So,  $V = nS_4 \oplus mS_5$ . But  $S_4, S_5$  are both 3-dimensional. So,  $V = S_4$  (or  $S_5$ ).

Using the fact that

$$\chi_1 + \chi_2 + 2\chi_3 + 3\chi_4 + 3\chi_5 = \chi_{reg}$$

we can now complete the character table of  $S_4$ :

$ c_j $	1	6	8	3	6
	1	(12)	(123)	(12)(34)	(1234)
$\chi_1$	1	1	1	1	1
$\chi_2$	1	-1	1	1	-1
$\chi_3$	2	0	-1	2	0
$\chi_4$	3	1	0	-1	-1
$\chi_5$	3	-1	0	-1	1

From the character table of  $S_4$  we can find all normal subgroups. First, the kernels of the 5 irreducible representations are:

- (1)  $\ker \rho_1 = S_4$ .
- (2)  $\ker \rho_2 = A_4$  containing the conjugacy classes of 1, (123), (12)(34).
- (3)  $\ker \rho_3 = K$  containing 1, (12)(34) and conjugates.
- (4)  $\ker \rho_4 = 1$ . I.e.,  $\rho_4$  is a faithful representation.
- (5)  $\ker \rho_5 = 1$ . So,  $\rho_5$  is also faithful.

Since these subgroups contain each other:

$$1 < K < A_4 < S_4$$

intersecting them will not give any other subgroups. So, these are the only normal subgroups of  $S_4$ .