Corollary 2.30. Suppose that the semisimple decomposition of the $G$-module $V$ is $V = \sum n_i S_i$. Then

$$n_i = \langle \chi_V, \chi_i \rangle$$

Proof. Since $\chi_{V \oplus W} = \chi_V + \chi_W$, we have: $\chi_V = \sum n_j \chi_j$. So,

$$\langle \chi_V, \chi_i \rangle = \langle \sum n_j \chi_j, \chi_i \rangle = n_i$$

\[\Box\]

2.5.2. Proof of the main theorem. The theorem will follow from three lemmas. The first lemma calculates the dimension of the fixed point set of $V$.

Definition 2.31. If $V$ is a $G$-module then the fixed point set of the action of $G$ is given by

$$V^G := \{ v \in V \mid \sigma v = v \ \forall \sigma \in G \}$$

Lemma 2.32. The dimension of the fixed point set is equal to the average value of the corresponding character:

$$\dim \mathbb{C} V^G = \frac{1}{n} \sum_{\sigma \in G} \chi_V(\sigma)$$

Proof. The projection map

$$\pi : V \to V^G$$

is given by

$$\pi(v) = \frac{1}{n} \sum \sigma v$$

It is clear that

1. $\pi(v) \in V^G$ since multiplication by any $\tau \in G$ will just permute the summands.
2. $\pi(v) = v$ if $v \in V^G$ because, in that case, each $\sigma v = v$ and there are $n$ terms.

Therefore, $\pi$ is a projection map, i.e., a linear retraction onto $V^G$. Looking at the formula we see that $\pi$ is multiplication by the idempotent $e_1 = \frac{1}{n} \sum_{\sigma \in G} \sigma$. (This is the idempotent corresponding to the trivial representation.) So:

$$\dim V^G = \text{Tr}(\pi) = \chi_V(e_1) = \chi_V \left( \frac{1}{n} \sum_{\sigma \in G} \sigma \right) = \frac{1}{n} \sum_{\sigma \in G} \chi_V(\sigma)$$

Explanations:
(1) \( \dim V^G = \Tr(\pi) \) because \( V \cong V^G \oplus W \) (\( W = \ker \pi \)). So, the matrix of \( \pi \) is:
\[
\pi = \begin{pmatrix} 1_{V^G} & 0 \\ 0 & 0_W \end{pmatrix}
\]
making \( \Tr(\pi) = \Tr(1_{V^G}) = \dim C V^G \).

(2) \( \Tr(\pi) = \chi_V(e_1) \) by definition of the character:
\[
\chi_V(e_1) := \Tr(e_1 \cdot : V \to V)
\]
This is the trace of the mapping \( V \to V \) given by multiplication by \( e_1 \).
But we are calling that mapping \( \pi \).

\[ \square \]

**Lemma 2.33.** If \( V, W \) are representations of \( G \) then
\[
\Hom_G(V, W) = \Hom_C(V, W)^G
\]
where \( G \) acts on \( \Hom_C(V, W) \) by conjugation, i.e., \( \sigma f = \sigma \circ f \circ \sigma^{-1} \)
which means that
\[
(\sigma f)(v) = \sigma f(\sigma^{-1}v)
\]

**Proof.** This is trivial. Given any linear map \( f : V \to W, f \) is a \( G \)-homomorphism iff
\[
\sigma \circ f = f \circ \sigma \iff \sigma \circ f \circ \sigma^{-1} = f \iff \sigma f = f
\]
iff \( f \in \Hom_C(V, W)^G \).

\[ \square \]

**Lemma 2.34.** \( \Hom_C(V, W) \cong V^* \otimes W \) as \( G \)-modules.

**Proof.** Let \( \phi : V^* \otimes W \to \Hom_C(V, W) \) be given by
\[
\phi(f \otimes w)(v) = f(v)w
\]
To check that this is a \( G \)-homomorphism we need to show that \( \phi \sigma = \sigma \phi \) for any \( \sigma \in G \). So, we compute both sides:
\[
\phi(\sigma f \otimes \sigma w)(v) = \phi(f \circ \sigma^{-1} \otimes \sigma w)
\]
which sends \( v \in V \) to
\[
\phi(f \circ \sigma^{-1} \otimes \sigma w)(v) = f(\sigma^{-1}v)\sigma w
\]
On the other side we have:
\[
\sigma \phi(f \otimes w)(v) = \sigma \circ \phi(f \otimes w) \circ \sigma^{-1}
\]
which also sends \( v \in V \) to
\[
\sigma \circ \phi(f \otimes w) \circ \sigma^{-1} v = \sigma(f(\sigma^{-1}v)w) = f(\sigma^{-1}v)\sigma w
\]
This shows that \( \phi \) commutes with the action of \( G \). The fact that \( \phi \) is an isomorphism is well-known: If \( v_i, v_i^* \) form a basis-dual basis pair for \( V \) and \( w_j \) form a basis for \( W \) then \( v_j^* \otimes w_i \) form a basis for \( V^* \otimes W \) and
\[
\phi(v_j^* \otimes w_i) : v = \sum a_j v_j \mapsto v_j^* (v) w_i = a_j w_i
\]
is the mapping whose matrix has $ij$-entry equal to 1 and all other entries 0. So, these homomorphisms form a basis for $\text{Hom}_C(V, W)$ and $\phi$ is an isomorphism.

Proof of main theorem 2.28. Using the three lemmas we get:

$$\dim_C \text{Hom}_G(V, W) = 2.33 \dim_C \text{Hom}_C(V, W)^G$$

$$= 2.34 \dim_C (V^* \otimes W)^G$$

$$= 2.32 \frac{1}{n} \sum_{\sigma \in G} \chi_{V^* \otimes W}(\sigma)$$

$$= \frac{1}{n} \sum_{\sigma} \chi_{V^*}(\sigma) \chi_W(\sigma)$$

$$= \frac{1}{n} \sum_{\sigma} \chi_V(\sigma^{-1}) \chi_W(\sigma) = \langle \chi_V, \chi_W \rangle$$

2.5.3. character table of $S_4$. Using these formulas we can calculate the character table for $S_4$. First note that there are five conjugacy classes represented by

$$1, (12), (123), (12)(34), (1234)$$

The elements of cycle form $(12)(34)$ form (with 1) a normal subgroup

$$K = \{1, (12)(34), (13)(24), (14)(23)\} \triangleleft S_4$$

called the Klein 4-group. The quotient $S_4/K$ is isomorphic to the symmetric group on 3 letters. Imitating the case of $D_4$, this allows us to construct the following portion of the character table for $S_4$:

| $|c_j|$ | 1 | 6 | 8 | 3 | 6 |
|------|---|---|---|---|---|
|      | 1 | (12) | (123) | (12)(34) | (1234) |

$$\begin{array}{cccc}
\chi_1 & 1 & 1 & 1 & 1 \\
\chi_2 & 1 & -1 & 1 & -1 \\
\chi_3 & 2 & 0 & -1 & 2 \\
\chi_4 & 3 & & & \\
\chi_5 & 3 & & & \\
\end{array}$$

Explanations:

(1) Since $(12)(34) \in K$, the value of the first three characters on this conjugacy class is $d_i$, the same as in the first column.

(2) Since $(1234)K = (12)K$, these two columns have the same values of $\chi_1, \chi_2, \chi_3$. 
(3) Finally, the two unknown characters \( \chi_4, \chi_5 \) must be 3-dimensional since

\[
24 = \sum d_i^2 = 1 + 1 + 4 + d_4^2 + d_5^2
\]

has only one solution: \( d_4 = d_5 = 3 \).

To figure out the unknown characters we need another representation. The permutation representation \( P \) is the 4-dimensional representation of \( S_4 \) in which the elements of \( S_4 \) act by permuting the unit coordinate vectors. For example

\[
\rho_P(12) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Note that the trace of \( \rho_P(\sigma) \) is equal to the number of letters left fixed by \( \sigma \). So, \( \chi_P \) takes values 4, 2, 1, 0, 0 as shown:

| \( |c_j| \) | 1 | 6 | 8 | 3 | 6 |
|-----|----|----|----|----|----|
|      | 1 (12) | (123) | (12)(34) | (1234) |
| \( \chi_1 \) | 1 | 1 | 1 | 1 | 1 |
| \( \chi_2 \) | 1 | -1 | 1 | 1 | -1 |
| \( \chi_3 \) | 2 | 0 | -1 | 2 | 0 |

\( \chi_V = \chi_P - \chi_1 \)

The representation \( P \) contains one copy of the trivial representation and no copies of the other two:

\[
\langle \chi_P, \chi_1 \rangle = \frac{1}{24} (4 + 6(2) + 8(1)) = 1
\]

\[
\langle \chi_P, \chi_2 \rangle = \frac{1}{24} (4 + 6(-1)(2) + 8(1)(1)) = 0
\]

\[
\langle \chi_P, \chi_3 \rangle = \frac{1}{24} ((2)(4) + 8(-1)(1)) = 0
\]

So, \( P \cong S_1 \oplus V \) where \( V \) is a 3-dimensional module which does not contain \( S_1, S_2 \) or \( S_3 \). So, \( V = nS_4 \oplus mS_5 \). But \( S_4, S_5 \) are both 3-dimensional. So, \( V = S_4 \) (or \( S_5 \)).

Using the fact that

\[
\chi_1 + \chi_2 + 2\chi_3 + 3\chi_4 + 3\chi_5 = \chi_{reg}
\]
we can now complete the character table of \( S_4 \):

\[
\begin{array}{c|cccccc}
|c_j| & 1 & 6 & 8 & 3 & 6 \\
\hline
& 1 & (12) & (123) & (12)(34) & (1234) \\
\hline
\chi_1 & 1 & 1 & 1 & 1 & 1 \\
\chi_2 & 1 & -1 & 1 & 1 & -1 \\
\chi_3 & 2 & 0 & -1 & 2 & 0 \\
\chi_4 & 3 & 1 & 0 & -1 & -1 \\
\chi_5 & 3 & -1 & 0 & -1 & 1 \\
\end{array}
\]

From the character table of \( S_4 \) we can find all normal subgroups. First, the kernels of the 5 irreducible representations are:

1. \( \ker \rho_1 = S_4 \).
2. \( \ker \rho_2 = A_4 \) containing the conjugacy classes of 1, (123), (12)(34).
3. \( \ker \rho_3 = K \) containing 1, (12)(34) and conjugates.
4. \( \ker \rho_4 = 1 \). I.e., \( \rho_4 \) is a faithful representation.
5. \( \ker \rho_5 = 1 \). So, \( \rho_5 \) is also faithful.

Since these subgroups contain each other:

\[
1 < K < A_4 < S_4
\]

intersecting them will not give any other subgroups. So, these are the only normal subgroups of \( S_4 \).
2.5.4. **column orthogonality.** The columns of the character table also satisfy an orthogonality condition. To see it we first have to write the row orthogonality condition

\[
\langle \chi_i, \chi_j \rangle = \sum_{k=1}^{b} \frac{|c_k|}{n} \chi_i(c_k) \overline{\chi_j(c_k)} = \delta_{ij}
\]

and write it in matrix form:

\[
T \begin{pmatrix} \frac{|c_1|}{n} & \cdots & 0 \\ 0 & \frac{|c_b|}{n} \end{pmatrix} \overline{T} = I_b
\]

where \( T \) is the character table \( T = (\chi_i(c_j)) \). This equation shows that the character table \( T \) is an invertible matrix with inverse

\[
T^{-1} = D \overline{T}
\]

where \( D \) is the diagonal matrix with diagonal entries \( \frac{|c_i|}{n} \). Multiplying both sides of this equation on the right by \( T \) and on the left with \( D^{-1} \) and we get:

\[
\overline{T} T = D^{-1} = \begin{pmatrix} \frac{n}{|c_1|} & 0 \\ 0 & \frac{n}{|c_b|} \end{pmatrix}
\]

Looking at the entries of these matrices we get the column orthogonality relation:

**Theorem 2.35.** If \( \sigma, \tau \in G \) then

\[
\sum_{i=1}^{b} \chi_i(\sigma) \overline{\chi_i(\tau)} = \begin{cases} \frac{n}{|c|} & \text{if } \sigma, \tau \text{ are conjugate} \\ 0 & \text{if not} \end{cases}
\]

Here \( |c| \) is the number of conjugates of \( \sigma \) in \( G \). (So, \( n/|c| \) is the order of the centralizer \( C(\sigma) = \{ \tau \in G \mid \sigma \tau = \tau \sigma \} \) of \( \sigma \).)

**Corollary 2.36.** The character table \( T = (\chi_i(c_j)) \) determines the size of each conjugacy class \( c_j \).

**Proof.** Taking \( \sigma = \tau \) in the above theorem we get

\[
|C(\sigma)| = \sum_i ||\chi_i(\sigma)||^2
\]

The size of the conjugacy class \( c \) of \( \sigma \) is the index of its centralizer: \( |c| = |G : C(\sigma)| = n/|C(\sigma)| \). \( \square \)
As an example, look at the character table for $S_3$:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>(12)</th>
<th>(123)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>2</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>

Column orthogonality means that the usual Hermitian dot product of the columns is zero. For example, the dot product of the first and third column is

$$(1)(1) + (1)(1) + (2)(-1) = 0$$

Also the dot product of the $j$th vector with itself (its length squared) is equal to $n/|c_j|$. For example, the length squared of the third column vector is

$$1 + 1 + 1 = 3$$

Making the number of conjugates of (123) equal to $6/3 = 2$. 
3. Induction

If \( H \) is a subgroup of \( G \) then any representation of \( G \) will restrict to a representation of \( H \) by composition:

\[
H \hookrightarrow G \xrightarrow{\rho} \text{Aut}_\mathbb{C}(V)
\]

Induction is a more complicated process which goes the other way: It starts with a representation of \( H \) and produces a representation of \( G \). Following Lang, I will construct the same object in several different ways starting with an elementary equation for the induced character.

3.1. induced characters.

**Definition 3.1.** Suppose that \( H \leq G \) (\( H \) is a subgroup of \( G \)) and \( \chi : H \rightarrow \mathbb{C} \) is a character (or any class function). Then the induced character

\[
\text{Ind}_H^G \chi : G \rightarrow \mathbb{C}
\]

is the class function on \( G \) defined by

\[
\text{Ind}_H^G \chi(\sigma) = \frac{1}{|H|} \sum_{\tau \in G} \chi(\tau \sigma \tau^{-1})
\]

where \( \chi(\sigma) = 0 \) if \( \sigma \notin H \).

The main theorem about the induced character is the following.

**Theorem 3.2.** If \( V \) is any representation of \( H \) then there exists a representation \( W \) of \( G \) so that

\[
\chi_W = \text{Ind}_H^G \chi_V
\]

Furthermore, \( W \) is unique up to isomorphism.

The representation \( W \) is written \( W = \text{Ind}_H^G V \) and is called the induced representation. We will study that tomorrow.

Before proving this theorem let me give two examples.

3.1.1. example 1. Here is a trivial observation.

**Proposition 3.3.** If \( G \) is abelian then

\[
\text{Ind}_H^G \chi(\sigma) = |G : H|\chi(\sigma)
\]

Now suppose that \( G = \mathbb{Z}/4 = \{1, \sigma, \sigma^2, \sigma^3\} \) and \( H = \{1, \tau\} \) with \( \tau = \sigma^2 \). Then the character table of \( H \cong \mathbb{Z}/2 \) is

\[
\begin{array}{ccc}
H = \mathbb{Z}/2 & 1 & \tau \\
\hline
\chi_+ & 1 & 1 \\
\chi_- & 1 & -1
\end{array}
\]