2.5.4. *column orthogonality.* The columns of the character table also satisfy an orthogonality condition. To see it we first have to write the row orthogonality condition

$$\langle \chi_i, \chi_j \rangle = \sum_{k=1}^{b} \frac{|c_k|}{n} \chi_i(c_k)\overline{\chi_j(c_k)} = \delta_{ij}$$

and write it in matrix form:

$$T \begin{pmatrix} \frac{|c_1|}{n} & & 0 \\ & \ddots & \\ 0 & & \frac{|c_b|}{n} \end{pmatrix} \overline{T}^t = I_b$$

where T is the character table $T = (\chi_i(c_j))$. This equation shows that the character table T is an invertible matrix with inverse

$$T^{-1} = D\overline{T}^t$$

where D is the diagonal matrix with diagonal entries $\frac{|c_i|}{n}$. Multiplying both sides of this equation on the right by T and on the left with D^{-1} and we get:

$$\overline{T}^t T = D^{-1} = \begin{pmatrix} \frac{n}{|c_1|} & & 0 \\ & \ddots & \\ 0 & & \frac{n}{|c_b|} \end{pmatrix}$$

Looking at the entries of these matrices we get the column orthogonality relation:

Theorem 2.35. *If $\sigma, \tau \in G$ then*

$$\sum_{i=1}^{b} \overline{\chi_i(\sigma)}\chi_i(\tau) = \begin{cases} \frac{n}{|c|} & \text{if } \sigma, \tau \text{ are conjugate} \\ 0 & \text{if not} \end{cases}$$

Here $|c|$ is the number of conjugates of σ in G. (So, $n/|c|$ is the order of the centralizer $C(\sigma) = \{\tau \in G \mid \sigma\tau = \tau\sigma\}$ *of σ.)*

Corollary 2.36. *The character table $T = (\chi_i(c_j))$ determines the size of each conjugacy class c_j.*

Proof. Taking $\sigma = \tau$ in the above theorem we get

$$|C(\sigma)| = \sum_i \|\chi_i(\sigma)\|^2$$

The size of the conjugacy class c of σ is the index of its centralizer: $|c| = |G : C(\sigma)| = n/|C(\sigma)|$. □

As an example, look at the character table for S_3:

	1	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Column orthogonality means that the usual Hermitian dot product of the columns is zero. For example, the dot product of the first and third column is

$$(1)(1) + (1)(1) + (2)(-1) = 0$$

Also the dot product of the jth vector with itself (its length squared) is equal to $n/|c_j|$. For example, the length squared of the third column vector is

$$1 + 1 + 1 = 3$$

Making the number of conjugates of (123) equal to $6/3 = 2$.