3. Induction

If H is a subgroup of G then any representation of G will restrict to a representation of H by composition:

$$H \hookrightarrow G \xrightarrow{\rho} \mathrm{Aut}_{\mathbb{C}}(V)$$

Induction is a more complicated process which goes the other way: It starts with a representation of H and produces a representation of G. Following Lang, I will construct the same object in several different ways starting with an elementary equation for the induced character.

3.1. induced characters.

Definition 3.1. *Suppose that $H \leq G$ (H is a subgroup of G) and $\chi : H \to \mathbb{C}$ is a character (or any class function). Then the* induced character

$$\mathrm{Ind}_H^G \chi : G \to \mathbb{C}$$

is the class function on G defined by

$$\mathrm{Ind}_H^G \chi(\sigma) = \frac{1}{|H|} \sum_{\tau \in G} \chi(\tau\sigma\tau^{-1})$$

where $\chi(\sigma) = 0$ if $\sigma \notin H$.

The main theorem about the induced character is the following.

Theorem 3.2. *If V is any representation of H then there exists a representation W of G so that*

$$\chi_W = \mathrm{Ind}_H^G \chi_V$$

Furthermore, W is unique up to isomorphism.

The representation W is written $W = \mathrm{Ind}_H^G V$ and is called the *induced representation*. We will study that tomorrow.

Before proving this theorem let me give two examples.

3.1.1. *example 1.* Here is a trivial observation.

Proposition 3.3. *If G is abelian then*

$$\mathrm{Ind}_H^G \chi(\sigma) = |G : H|\chi(\sigma)$$

Now suppose that $G = \mathbb{Z}/4 = \{1, \sigma, \sigma^2, \sigma^3\}$ and $H = \{1, \tau\}$ with $\tau = \sigma^2$. Then the character table of $H \cong \mathbb{Z}/2$ is

$H = \mathbb{Z}/2$	1	τ
χ_+	1	1
χ_-	1	-1

I want to calculate $\operatorname{Ind}_{\mathbb{Z}/2}^{\mathbb{Z}/4} \chi_-$. By the proposition, the value of this induced character on $1, \sigma, \sigma^2, \sigma^3$ is the index $|G : H| = 2$ times $1, 0, -1, 0$ respectively. This gives $2, 0, -2, 0$ as indicated below the character table for $G = \mathbb{Z}/4$:

$G = \mathbb{Z}/4$	1	σ	σ^2	σ^3
χ_1	1	1	1	1
χ_2	1	-1	1	-1
χ_3	1	i	-1	$-i$
χ_4	1	$-i$	-1	i
$\operatorname{Ind}_{\mathbb{Z}/2}^{\mathbb{Z}/4} \chi_-$	2	0	-2	0

By examination we see that

$$\operatorname{Ind}_{\mathbb{Z}/2}^{\mathbb{Z}/4} \chi_- = \chi_3 + \chi_4$$

3.1.2. *example 2.* In the nonabelian case we have the following formula which is analogous to the one in the abelian case.

Proposition 3.4.

$$\operatorname{Ind}_H^G \chi(\sigma) = |G : H|(\textit{average value of } \chi(\tau\sigma\tau^{-1}))$$

Now let $G = S_3$ and $H = \{1, (12)\} \cong \mathbb{Z}/2$. Using the same notation as in the previous example, let χ_- be the one dimensional character on H given by $\chi_-(1) = 1, \chi_-(12) = -1$. We want to compute the induced character $\operatorname{Ind}_H^G \chi_-$.

$$\operatorname{Ind}_H^G \chi_-(1) = |G : H|\chi_-(1) = (3)(1) = 3$$

Since (12) has three conjugates only one of which lies in H, the average value of χ_1 on these conjugates is

$$\frac{1}{3}(-1 + 0 + 0) = -\frac{1}{3}$$

So,

$$\operatorname{Ind}_H^G \chi_-(12) = |G : H| \left(\frac{-1}{3} \right) = \frac{-3}{3} = -1$$

Since neither of the conjugates of (123) lie in H we have:

$$\operatorname{Ind}_H^G \chi_-(123) = 0$$

So, $\operatorname{Ind}_H^G \chi_-$ takes the values $3, -1, 0$ on the conjugacy classes of $G = S_3$. Put it below the character table of S_3:

$G = S_3$	1	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1
$\operatorname{Ind}_H^G \chi_-$	3	-1	0

We can see that

$$\operatorname{Ind}_H^G \chi_- = \chi_2 + \chi_3$$

3.1.3. *Frobenius reciprocity for characters.* First I need some fancy notation for a very simple concept. If $f : G \to \mathbb{C}$ is any class function then the *restriction* of f to H, denoted $\operatorname{Res}_H^G f$, is the composition of f with the inclusion map $j : H \hookrightarrow G$:

$$\operatorname{Res}_H^G f = f \circ j : H \to \mathbb{C}$$

Theorem 3.5 (Frobenius reciprocity). *Suppose that g, h are class functions on G, H respectively. Then*

$$\left\langle \operatorname{Ind}_H^G h, g \right\rangle_G = \left\langle h, \operatorname{Res}_H^G g \right\rangle_H$$

Suppose for a moment that this is true. Then, letting $h = \chi_V$ and taking g to be the irreducible character $g = \chi_i$, we get:

$$\left\langle \operatorname{Ind}_H^G \chi_V, \chi_i \right\rangle_G = \left\langle \chi_V, \operatorname{Res}_H^G \chi_i \right\rangle_H = n_i$$

Since $\operatorname{Res}_H^G \chi_i$ is the character of the G-module S_i considered as an H-module, the number n_i is a nonnegative integer, namely:

$$n_i = \dim_{\mathbb{C}} \operatorname{Hom}_H(V, S_i)$$

This implies that

$$\operatorname{Ind}_H^G \chi_V = \chi_W$$

where W is the G-module $W = \bigoplus n_i S_i$. In other words, the induced character is an *effective character* (the character of some representation).

Corollary 3.6. *If $h : H \to \mathbb{C}$ is an effective character then so is $\operatorname{Ind}_H^G h : G \to \mathbb{C}$.*

This is a rewording of the main theorem (Theorem **??**).

Proof of Frobenius reciprocity for characters. Since

$$\operatorname{Ind}_H^G h(\sigma) = \frac{1}{|H|}\sum_{\tau\in G} h(\tau\sigma\tau^{-1})$$

the left hand side of our equation is

$$LHS = \frac{1}{|G|}\sum_{\sigma\in G}\frac{1}{|H|}\sum_{\tau\in G} h(\tau\sigma\tau^{-1})g(\sigma^{-1})$$

Since g is a class function, $g(\sigma^{-1}) = g(\tau\sigma^{-1}\tau^{-1})$. Letting $\alpha = \tau\sigma\tau^{-1}$ we get a sum of terms of the form

$$h(\alpha)g(\alpha^{-1})$$

How many times does each such term occur?

Claim: The number of ways that α can be written as $\alpha = \tau\sigma\tau^{-1}$ is exactly $n = |G|$.

The proof of this claim is simple. For each $\tau \in G$ there is exactly one σ which works, namely, $\sigma = \tau^{-1}\alpha\tau$.

This implies that

$$LHS = \frac{1}{|H|}\sum_{\alpha\in G} h(\alpha)g(\alpha^{-1})$$

Since h is a class function on H, $h(\alpha) = 0$ if $\alpha \notin H$. Therefore, the sum can be restricted to $\alpha \in H$ and this expression is equal to the RHS of the Frobenius reciprocity equation. □

3.1.4. *examples of Frobenius reciprocity.* Let's take the two example of induced characters that we did earlier and look at what Frobenius reciprocity says about them.

In the case $G = \mathbb{Z}/4, H = \mathbb{Z}/2$, the restrictions of the four irreducible characters of $G = \mathbb{Z}/4$ to H (given by the first and third columns) are:

$$\operatorname{Res}_H^G \chi_1 = \chi_+$$
$$\operatorname{Res}_H^G \chi_2 = \chi_+$$
$$\operatorname{Res}_H^G \chi_3 = \chi_-$$
$$\operatorname{Res}_H^G \chi_4 = \chi_-$$

Frobenius reciprocity says that the number of times that χ_- appears in the decomposition of $\operatorname{Res}_H^G \chi_i$ is equal to the number of times that χ_i appears in the decomposition of $\operatorname{Ind}_H^G \chi_-$. So,

$$\operatorname{Ind}_H^G \chi_- = \chi_3 + \chi_4$$

In the case $G = S_3, H = \{1, (12)\}$, the restrictions of the three irreducible characters of $G = S_3$ to H, as given by the first two columns, are:

$$\mathrm{Res}_H^G \chi_1 = \chi_+$$
$$\mathrm{Res}_H^G \chi_2 = \chi_-$$
$$\mathrm{Res}_H^G \chi_3 = (2, 0) = \chi_+ + \chi_-$$

Since χ_- appears once in the restrictions of χ_2, χ_3 we have

$$\mathrm{Ind}_H^G \chi_- = \chi_2 + \chi_3$$

3.1.5. *induction-restriction tables.* The results of the calculations in these two examples are summarized in the following tables which are called *induction-restriction tables.*

For $G = \mathbb{Z}/4$ and $H = \mathbb{Z}/2$ the induction-restriction table is:

	χ_+	χ_-
χ_1	1	0
χ_2	1	0
χ_3	0	1
χ_4	0	1

For $G = S_3$ and $H = \{1, (12)\}$ the induction-restriction table is:

	χ_+	χ_-
χ_1	1	0
χ_2	0	1
χ_3	1	1

In both cases, the rows give the decompositions of $\mathrm{Res}_H^G \chi_i$ and the columns give the decompositions of $\mathrm{Ind}_H^G \chi_\pm$.