(b) f is *balanced* in the sense that

$$f(xs, v) = f(x, sv)$$

for all $x \in M, s \in S, v \in V$.

In other words, f is bilinear and balanced.

(2) For any other bilinear, balanced mapping $g : M \times V \to W$ there is a unique homomorphism $\widehat{g} : M \otimes_S V \to W$ so that $g = \widehat{g} \circ f$

Let $\mathrm{BiLin}(M \times_S V, W)$ denote the set of all balanced bilinear maps $M \times V \to W$. Then the universal property says that

$$\mathrm{BiLin}(M \times_S V, W) \cong \mathrm{Hom}(M \otimes_S V, W)$$

On the other hand the definitions of balanced and bilinear imply that

$$\mathrm{BiLin}(M \times_S V, W) \cong \mathrm{Hom}_S(V, \mathrm{Hom}(M, W))$$

The balance bilinear map $\phi : M \times_S V \to W$ corresponds to its adjoint $\widehat{\phi} : V \to \mathrm{Hom}(M, W)$ given by $\widehat{\phi}(v)(x) = \phi(x, v)$.

(1) $\phi(x, v)$ is linear in x iff $\widehat{\phi}(v) \in \mathrm{Hom}(M, W)$. This is clear.
(2) $\phi(x, v)$ is linear in v iff $\widehat{\phi}$ is linear, i.e., gives a homomorphism of abelian groups $V \to \mathrm{Hom}(M, W)$. This is also clear.
(3) Finally, ϕ is balance iff $\phi(xs, v) = \phi(x, sv)$ iff

$$\widehat{\phi}(sv)(x) = \widehat{\phi}(v)(xs) = [s\widehat{\phi}(v)](x)$$

iff $\widehat{\phi}s = s\widehat{\phi}$.

In the case when M is an R-S-bimodule we just need to observe the obvious fact that ϕ is an R-homomorphism in the first coordinate iff $\widehat{\phi}(V) \subseteq \mathrm{Hom}_R(M, W)$.

The adjunction formula follows from these observations. □

3.2.2. *example.* Here is the simplest example of an induced representation. Take $G = \mathbb{Z}/4 = \{1, \tau, \tau^2, \tau^3\}$ and $H = \mathbb{Z}/2 = \{1, \sigma\}$ where $\sigma = \tau^2$. Let ρ be the one dimensional sign representation $\rho(\sigma) = -1$. Let V denote the H-module of the representation. So, $H = \mathbb{C}$ with σ acting by -1.

What is the induced representation $\mathrm{Ind}_{\mathbb{Z}/2}^{\mathbb{Z}/4} \rho$?

The induced module is $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$ which is 2-dimensional. It is generated by four elements $1 \otimes 1, \tau \otimes 1, \tau^2 \otimes 1, \tau^3 \otimes 1$. But $\tau^2 = \sigma$. So,

$$\tau^2 \otimes 1 = 1 \otimes \sigma 1 = -1 \otimes 1$$

and

$$\tau^3 \otimes 1 = \tau \otimes \sigma 1 = -\tau \otimes 1$$

So, $\mathbb{C}[G] \otimes V$ is two dimensional with basis $w_1 = 1 \otimes 1, w_2 = \tau \otimes 1$ and τ acts by: $\tau w_1 = w_2$ and $\tau w_2 = \tau^2 \otimes 1 = -1 \otimes 1 = -w_1$. So, the matrix of the representation $\operatorname{Ind}_H^G \rho = \phi$ is given by:

$$\phi(\tau) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Since G is cyclic this determines the other matrices:

$$\phi(\tau^2) = \phi(\tau)^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \phi(\tau^3) = \phi(\tau)^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Notice that matrices are all "monomial" which means that they have exactly one nonzero entry in every row and every column. The induced representation is always given by monomial matrices.

3.2.3. *monomial matrices.* A *monomial matrix* of size m with coefficients in a group H is defined to be an element of $\mathrm{Mat}_m(\mathbb{Z}[H])$ having exactly one nonzero entry in every row and every column and so that those entries lie in H. Every monomial matrix M is a product of a permutation matrix P_σ and a diagonal matrix D:

$$M = P_\sigma D(h_1, h_2, \cdots, h_m)$$

Here P_σ is the matrix obtained from the identity matrix I_m by permuting the rows by the permutation σ. For example, if $\sigma = (132)$ then

$$P_{(132)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

This is obtained by taking the identity matrix, moving row 1 which is $(1, 0, 0)$ to row $\sigma(1) = 3$, moving row 2 which is $(0, 1, 0)$ to row $\sigma(2) = 1$, etc. The entries of the matrix are:

$$(P_\sigma)_{ij} = \begin{cases} 1 & \text{if } i = \sigma(j) \\ 0 & \text{otherwise} \end{cases}$$

The notation for the diagonal group is the obvious one: $D(h_1, \cdots, h_m)$ is the diagonal matrix with (i, i) entry h_i. So, for example,

$$P_{(132)} D(h_1, h_2, h_3) = \begin{pmatrix} 0 & h_2 & 0 \\ 0 & 0 & h_3 \\ h_1 & 0 & 0 \end{pmatrix}$$

So, h_j is in the jth column.

How do monomial matrices multiply? We need to calculate:

$$P_\sigma D(h_1, \cdots, h_m) P_\tau D(\ell_1, \cdots, \ell_m)$$

But

$$D(h_1, \cdots, h_m)P_\tau = P_\tau D(h_{\tau(1)}, \cdots, h_{\tau(m)})$$

So,

$$P_\sigma D(h_1, \cdots, h_m)P_\tau D(\ell_1, \cdots, \ell_m) = P_{\sigma\tau} D(h_{\tau(1)}\ell_1, \cdots, h_{\tau(m)}\ell_m) \tag{3.1}$$

Definition 3.11. Let $M_m(H)$ denote the group of all $m \times m$ monomial matrices with coefficients in H. We denote the elements by

$$M(\sigma; h_1, \cdots, h_m) = P_\sigma D(h_1, \cdots, h_m)$$

3.2.4. *monomial representation.* Suppose that H is a subgroup of a group G with index $|G : H| = m$. Then

$$G = t_1 H \cup t_2 H \cup \cdots \cup t_m H$$

where $t_1, \cdots, t_m$ form what is called a (left) *transversal* which is a set of representatives for the left cosets of H. Then we will get a *monomial representation* by which I mean a homomorphism

$$\rho : G \to M_m(H)$$

First, I start with the *permutation representation*

$$\pi : G \to S_m$$

which is given by the action of G on the set of left cosets of H. If $\sigma \in G$ then

$$\sigma t_j H = t_i H$$

where $i = \sigma(j) = \pi(\sigma)(j)$.

For example, suppose $G = S_3, H = \{1, (12)\}$. Choose the transversal: $t_1 = 1, t_2 = (13), t_3 = (23)$. Then $\sigma = (13)$ acts on the three left cosets by transposing the first two and fixing the third:

$$(13)t_1 H = t_2 H, \quad (13)t_2 H = t_1 H, \quad (13)t_3 H = t_3 H$$

Therefore, $\pi(13) = (12)$.

Now, look at the element of H that we get:

$$\sigma t_j = t_{\sigma(j)} h_j$$

where

$$\boxed{h_j = t_{\sigma(j)}^{-1} \sigma t_j}$$

Definition 3.12. The *monomial representation*

$$\rho : G \to M_m(H)$$

is given by

$$\rho(\sigma) = M(\pi(\sigma); t_{\sigma(1)}^{-1}\sigma t_1, \cdots, t_{\sigma(m)}^{-1}\sigma t_m)$$

The following calculation verifies that ρ is a homomorphism:

$$\rho(\sigma)\rho(\tau) = M(\pi(\sigma); t_{\sigma(1)}^{-1}\sigma t_1, \cdots, t_{\sigma(m)}^{-1}\sigma t_m) M(\pi(\tau); t_{\tau(1)}^{-1}\tau t_1, \cdots, t_{\tau(m)}^{-1}\tau t_m)$$

$$= M(\pi(\sigma)\pi(\tau); \cdots, \left(t_{\sigma(i)}^{-1}\sigma t_i\right)\left(t_{\tau(j)}^{-1}\tau t_j\right), \cdots)$$

But $i = \tau(j)$ by the formula (3.1). So,

$$\left(t_{\sigma(i)}^{-1}\sigma t_i\right)\left(t_{\tau(j)}^{-1}\tau t_j\right) = t_{\sigma\tau(j)}^{-1}\sigma\tau t_j$$

and

$$\rho(\sigma)\rho(\tau) = M(\pi(\sigma\tau); \cdots, t_{\sigma\tau(j)}^{-1}\sigma\tau t_j, \cdots) = \rho(\sigma\tau)$$

3.2.5. *induced representation as monomial representation.* Suppose that $\phi : H \to GL(k, \mathbb{C})$ is a k-dimensional representation of H and $V \cong \mathbb{C}^k$ is the corresponding H-module. Then I claim that the induced representation $\operatorname{Ind}_H^G \phi$ is a monomial representation. More precisely the statement is:

Proposition 3.13. *The induced representation*

$$\psi = \operatorname{Ind}_H^G \phi : G \to GL(mk, \mathbb{C})$$

is the composition of the monomial representation $\rho : G \to M_m(H)$ with the homomorphism

$$M_m(\phi) : M_m(H) \to M_m(GL(k, \mathbb{C})) \subseteq GL(mk, \mathbb{C})$$

induced by $\phi : H \to GL(k, \mathbb{C})$.

Proof. As a right H-module, $\mathbb{C}[G]$ is free of rank m with a basis given by a left transversal $t_1, \cdots, t_m$. So,

$$\mathbb{C}[G] \cong t_1\mathbb{C}[H] \oplus \cdots \oplus t_m\mathbb{C}[H]$$

As a G-module the induced representation is defined to be

$$\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V = (t_1 \otimes V) \oplus \cdots \oplus (t_m \otimes V)$$

An arbitrary element is given by $\sum_j t_j \otimes v_j$ where v_j are arbitrary elements of V. Each $\sigma \in G$ acts by

$$\sigma \sum t_j \otimes v_j = \sum \sigma t_j \otimes v_j = \sum t_{\sigma(j)} h_j \otimes v_j = \sum t_{\sigma(j)} \otimes \phi(h_j) v_j$$

In other words, σ acts on V^m by multiplying the jth copy of V by the matrix

$$\phi(h_j) = \phi(t_{\sigma(j)}^{-1}\sigma t_j)$$

and then moving it to the $\sigma(j)$ slot. So:

$$\operatorname{Ind}_H^G \phi = M(\pi(\sigma); \cdots, \phi(t_{\sigma(j)}^{-1}\sigma t_j), \cdots)$$

This is $M_m(\phi)$ applied to the standard monomial representation as I claimed. □

Proposition 3.14. *The character of the induced representation is the induced character.*

Proof. This is a simple calculation. The trace of a monomial matrix is given by the points left fixed by the permutation representation $\pi(\sigma)$:

$$\begin{aligned}\mathrm{Tr}(\mathrm{Ind}_H^G \phi) &= \mathrm{Tr}\, M(\pi(\sigma); \cdots, \phi(t_{\sigma(j)}^{-1}\sigma t_j), \cdots) \\ &= \sum_{j=\sigma(j)} \mathrm{Tr}\, \phi(t_{\sigma(j)}^{-1}\sigma t_j) = \sum_{j=1}^{m} \chi_\phi(t_j^{-1}\sigma t_j)\end{aligned}$$

because $\chi_\phi(t_j^{-1}\sigma t_j) = 0$ when $j \neq \sigma(j)$.

Since χ_ϕ is a class function on H,

$$\chi_\phi(t_j^{-1}\sigma t_j) = \chi_\phi(h^{-1}t_j^{-1}\sigma t_j h)$$

for all $h \in H$. So,

$$\mathrm{Tr}(\mathrm{Ind}_H^G \phi) = \frac{1}{|H|} \sum_{h \in H} \sum_{j=1}^{m} \chi_\phi(h^{-1}t_j^{-1}\sigma t_j h)$$

Since $t_j h$ runs over all the elements of G, this is equal to

$$\mathrm{Ind}_H^G \chi_\phi(\sigma) = \frac{1}{|H|} \sum_{\tau \in G} \chi_\phi(\tau^{-1}\sigma\tau)$$

proving the proposition. □

3.3. **Artin's theorem.** One of the main theorems is that all characters on finite groups are integer linear combinations of characters induced from abelian subgroups. I don't have time to do this theorem. But I can prove a weaker version which says that all characters are rational linear combinations of characters induced from cyclic subgroups.

Before I prove this, I want to make sense out of the statement of the theorem. What happens when we take linear combinations of characters when the coefficients are arbitrary integers or rational numbers?

3.3.1. *character ring.*

Definition 3.15. The *character ring* $R(G)$ of G is defined to be the ring of all *virtual characters* which are defined to be differences of effective characters:

$$f = \chi_V - \chi_W$$

These can also be described as integer linear combination of irreducible characters:

$$f = \sum n_i \chi_i, \quad n_i \in \mathbb{Z}$$

$R(G)$ is a ring because (pointwise) sums and products of effective characters are effective. So, the same holds for virtual characters.

Proposition 3.16. *A group homomorphism $\phi : H \to G$ induces a ring homomorphism $\phi^* : R(G) \to R(H)$. In particular, if $H \leq G$,*

$$\operatorname{Res}_H^G : R(G) \to R(H)$$

is a ring homomorphism.

I won't prove this because it is sort of obvious and I don't need it. I want to look at the induction map.

Proposition 3.17. *If $H \leq G$ then*

$$\operatorname{Ind}_H^G : R(H) \to R(G)$$

is a group homomorphism, i.e., it is additive.

Proof. This follows from the fact that tensor product distributes over direct sum:

$$\begin{aligned}
\operatorname{Ind}_H^G(V \oplus W) &= \mathbb{C}[G] \otimes_{\mathbb{C}[H]} (V \oplus W) \\
&\cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V \oplus \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W \\
&= \operatorname{Ind}_H^G V \oplus \operatorname{Ind}_H^G W
\end{aligned}$$

□