

3.3.2. *statement of the theorem.* We want a collection of subgroups  $\mathcal{X} = \{H\}$  of  $G$  with the property that the maps  $\text{Ind}_H^G : R(H) \rightarrow R(G)$  taken together for all  $H \in \mathcal{X}$  give an epimorphism

$$\sum \text{Ind}_H^G : \bigoplus_{H \in \mathcal{X}} R(H) \rightarrow R(G)$$

This would say that every (effective) character on  $G$  is an integer linear combination of characters induced from the subgroups  $H \in \mathcal{X}$ . But we will only get this rationally which is the same as saying that the cokernel is a finite group.

**Theorem 3.18** (Artin). *Suppose that  $\mathcal{X}$  is a collection of subgroups  $H \leq G$ . Then the following conditions are equivalent.*

- (1)  $\forall \sigma \in G \exists H \in \mathcal{X}$  so that  $H$  contains a conjugate of  $\sigma$ .
- (2) Every character on  $G$  is a rational linear combination of characters induced from the subgroups  $H \in \mathcal{X}$ .

As an example, the collection of cyclic subgroups of  $G$  satisfies condition (1) since every element of  $G$  is contained in a cyclic subgroup.

3.3.3. *example:  $D_4$ .* Take the dihedral group

$$G = D_4 = \{1, \sigma, \sigma^2, \sigma^3, \tau, \tau\sigma, \tau\sigma^2, \tau\sigma^3\}$$

Let  $\mathcal{X} = \{\mathbb{Z}/4, \langle \tau \rangle, \langle \tau\sigma \rangle\}$ . These three subgroups meet all of the conjugacy classes of  $D_4$ . So, Artin's theorem applies. To find the image of the induction map we start with the character table of  $D_4$ :

	1	$\sigma^2$	$\sigma$	$\tau$	$\sigma\tau$
$\chi_1$	1	1	1	1	1
$\chi_2$	1	1	-1	1	-1
$\chi_3$	1	1	-1	-1	1
$\chi_4$	1	1	1	-1	-1
$\chi_5$	2	-2	0	0	0

From this we can easily compute the induction-restriction table:

$D_4$	$\mathbb{Z}/4$				$\langle \tau \rangle$		$\langle \tau\sigma \rangle$	
	$\chi_1$	$\chi_i$	$\chi_{-1}$	$\chi_{-i}$	$\chi_+$	$\chi_-$	$\chi_+$	$\chi_-$
$\chi_1$	1				1		1	
$\chi_2$			1		1			1
$\chi_3$			1			1	1	
$\chi_4$	1					1		1
$\chi_5$		1		1	1	1	1	1

Here  $\chi_\xi$  denotes the one dimensional character of a cyclic group of order  $n$  which sends the generator to  $\xi$  (which must be an  $n$ th root of unity).

This  $5 \times 8$  matrix  $T$  gives the induction map:

$$\begin{aligned} R(\mathbb{Z}/4) \oplus R(\langle \tau \rangle) \oplus R(\langle \tau\sigma \rangle) &\xrightarrow{\text{Ind}} R(D_4) \\ \mathbb{Z}^4 \oplus \mathbb{Z}^2 \oplus \mathbb{Z}^2 &\xrightarrow{\text{multiplication by } T} \mathbb{Z}^5 \end{aligned}$$

Artin's theorem says that the cokernel of this map is a finite group. To find this group we use integer row and column operations, which change the basis for  $\mathbb{Z}^5$  and  $\mathbb{Z}^8$  respectively, to reduce the matrix  $T$  to the form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \end{pmatrix}$$

This means that the cokernel of the induction map is  $\mathbb{Z}/2$ . So, for any representation  $V$  of  $D_4$ , twice the character of  $V$  is a sum of virtual characters induced from virtual representations of the three cyclic subgroup in the list  $\mathcal{X}$ .

3.3.4. *proof of the theorem.* (2)  $\Rightarrow$  (1). Let  $\sigma \in G$ . Then there is an irreducible character  $\chi_i$  so that  $\chi_i(\sigma) \neq 0$ . Since  $\chi_i$  is a rational linear combination of induced characters from  $H \in \mathcal{X}$ , there must be some  $H \in \mathcal{X}$  and some representation  $V$  of  $H$  so that  $\text{Ind}_H^G \chi_V(\sigma) \neq 0$ . By the definition of induced character this implies that some conjugate of  $\sigma$  lies in  $H$ .

(1)  $\Rightarrow$  (2). Suppose that (2) is false. Then the set of induced virtual characters forms a subgroup  $L$  of  $R(G) \cong \mathbb{Z}^b$  of rank  $a < b$ . Let  $\phi_1, \dots, \phi_a$  be a set of characters induced from elements  $H \in \mathcal{X}$  which span  $L$ . We can decompose each  $\phi_i$  into an integer linear combination of the irreducible characters  $\chi_j$ :

$$\phi_i = \sum n_{ij} \chi_j$$

The numbers  $n_{ij}$  form an  $a \times b$  matrix which defines a  $\mathbb{Q}$  linear map:

$$(n_{ij}) : \mathbb{Q}^b \rightarrow \mathbb{Q}^a$$

Since  $a < b$  this linear map has a kernel, i.e., there are rational numbers  $c_j$  not all zero so that

$$\sum_j n_{ij} c_j = 0 \quad \forall i$$

Multiplying by the denominators, we may assume the numbers  $c_j$  are integers. This gives a nonzero virtual character

$$\sum c_j \chi_j = \chi_V - \chi_W$$

which is orthogonal to all the  $\phi_i$  and therefore all  $\phi \in L$ :

$$\langle \phi_i, \chi_W - \chi_{W'} \rangle = \left\langle \phi_i, \sum c_j \chi_j \right\rangle = \sum n_{ij} c_j = 0$$

But  $L$  contains all induced characters:

$$\phi = \text{Ind}_H^G V$$

for all  $H \in \mathcal{X}$  and all representations  $V$  of  $H$ . So, by Frobenius reciprocity, we have:

$$\langle \phi, \chi_W - \chi_{W'} \rangle = \langle \text{Ind}_H^G V, \chi_W - \chi_{W'} \rangle_G = \langle V, \text{Res}_H^G(\chi_W - \chi_{W'}) \rangle_H = 0$$

Since this is true for all representations  $V$  of  $H$ , we must have

$$\text{Res}_H^G(\chi_W - \chi_{W'}) = 0$$

for all  $H \in \mathcal{X}$ . This in turn implies that

$$\chi_W(\sigma) = \chi_{W'}(\sigma)$$

for all  $\sigma \in H$ .

But, for any  $\sigma \in G$  there is an  $H \in \mathcal{X}$  which contains a conjugate of  $\sigma$ . But then

$$\chi_W(\sigma) = \chi_{W'}(\sigma)$$

So, the virtual character  $\chi_W - \chi_{W'}$  must be zero, which is a contradiction. This proves the theorem.  $\square$