

3.3. Artin's theorem. One of the main theorems is that all characters on finite groups are integer linear combinations of characters induced from abelian subgroups. I don't have time to do this theorem. But I can prove a weaker version which says that all characters are rational linear combinations of characters induced from cyclic subgroups.

Before I prove this, I want to make sense out of the statement of the theorem. What happens when we take linear combinations of characters when the coefficients are arbitrary integers or rational numbers?

3.3.1. character ring.

Definition 3.15. The character ring $R(G)$ of G is defined to be the ring of all virtual characters which are defined to be differences of effective characters:

$$f = \chi_V - \chi_W$$

These can also be described as integer linear combination of irreducible characters:

$$f = \sum n_i \chi_i, \quad n_i \in \mathbb{Z}$$

$R(G)$ is a ring because (pointwise) sums and products of effective characters are effective. So, the same holds for virtual characters.

Proposition 3.16. A group homomorphism $\phi : H \rightarrow G$ induces a ring homomorphism $\phi^* : R(G) \rightarrow R(H)$. In particular, if $H \leq G$,

$$\text{Res}_H^G : R(G) \rightarrow R(H)$$

is a ring homomorphism.

I won't prove this because it is sort of obvious and I don't need it. I want to look at the induction map.

Proposition 3.17. If $H \leq G$ then

$$\text{Ind}_H^G : R(H) \rightarrow R(G)$$

is a group homomorphism, i.e., it is additive.

Proof. This follows from the fact that tensor product distributes over direct sum:

$$\begin{aligned} \text{Ind}_H^G(V \oplus W) &= \mathbb{C}[G] \otimes_{\mathbb{C}[H]} (V \oplus W) \\ &\cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V \oplus \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W \\ &= \text{Ind}_H^G V \oplus \text{Ind}_H^G W \end{aligned}$$

□

3.3.2. *statement of the theorem.* We want a collection of subgroups $\mathcal{X} = \{H\}$ of G with the property that the maps $\text{Ind}_H^G : R(H) \rightarrow R(G)$ taken together for all $H \in \mathcal{X}$ give an epimorphism

$$\sum \text{Ind}_H^G : \bigoplus_{H \in \mathcal{X}} R(H) \twoheadrightarrow R(G)$$

This would say that every (effective) character on G is an integer linear combination of characters induced from the subgroups $H \in \mathcal{X}$. But we will only get this rationally which is the same as saying that the cokernel is a finite group.

Theorem 3.18 (Artin). *Suppose that \mathcal{X} is a collection of subgroups $H \leq G$. Then the following conditions are equivalent.*

- (1) $\forall \sigma \in G \exists H \in \mathcal{X}$ so that H contains a conjugate of σ .
- (2) Every character on G is a rational linear combination of characters induced from the subgroups $H \in \mathcal{X}$.

As an example, the collection of cyclic subgroups of G satisfies condition (1) since every element of G is contained in a cyclic subgroup.

3.3.3. *example: D_4 .* Take the dihedral group

$$G = D_4 = \{1, \sigma, \sigma^2, \sigma^3, \tau, \tau\sigma, \tau\sigma^2, \tau\sigma^3\}$$

Let $\mathcal{X} = \{\mathbb{Z}/4, \langle \tau \rangle, \langle \tau\sigma \rangle\}$. These three subgroups meet all of the conjugacy classes of D_4 . So, Artin's theorem applies. To find the image of the induction map we start with the character table of D_4 :

	1	σ^2	σ	τ	$\sigma\tau$
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	-1	-1	1
χ_4	1	1	1	-1	-1
χ_5	2	-2	0	0	0

From this we can easily compute the induction-restriction table:

D_4	$\mathbb{Z}/4$				$\langle \tau \rangle$		$\langle \tau\sigma \rangle$	
	χ_1	χ_i	χ_{-1}	χ_{-i}	χ_+	χ_-	χ_+	χ_-
χ_1	1				1		1	
χ_2			1		1			1
χ_3			1			1	1	
χ_4	1					1		1
χ_5		1		1	1	1	1	1

Here χ_ξ denotes the one dimensional character of a cyclic group of order n which sends the generator to ξ (which must be an n th root of unity).

This 5×8 matrix T gives the induction map:

$$\begin{aligned} R(\mathbb{Z}/4) \oplus R(\langle \tau \rangle) \oplus R(\langle \tau\sigma \rangle) &\xrightarrow{\text{Ind}} R(D_4) \\ \mathbb{Z}^4 \oplus \mathbb{Z}^2 \oplus \mathbb{Z}^2 &\xrightarrow{\text{multiplication by } T} \mathbb{Z}^5 \end{aligned}$$

Artin's theorem says that the cokernel of this map is a finite group. To find this group we use integer row and column operations, which change the basis for \mathbb{Z}^5 and \mathbb{Z}^8 respectively, to reduce the matrix T to the form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \end{pmatrix}$$

This means that the cokernel of the induction map is $\mathbb{Z}/2$. So, for any representation V of D_4 , twice the character of V is a sum of virtual characters induced from virtual representations of the three cyclic subgroup in the list \mathcal{X} .

3.3.4. *proof of the theorem.* (2) \Rightarrow (1). Let $\sigma \in G$. Then there is an irreducible character χ_i so that $\chi_i(\sigma) \neq 0$. Since χ_i is a rational linear combination of induced characters from $H \in \mathcal{X}$, there must be some $H \in \mathcal{X}$ and some representation V of H so that $\text{Ind}_H^G \chi_V(\sigma) \neq 0$. By the definition of induced character this implies that some conjugate of σ lies in H .

(1) \Rightarrow (2). Suppose that (2) is false. Then the set of induced virtual characters forms a subgroup L of $R(G) \cong \mathbb{Z}^b$ of rank $a < b$. Let ϕ_1, \dots, ϕ_a be a set of characters induced from elements $H \in \mathcal{X}$ which span L . We can decompose each ϕ_i into an integer linear combination of the irreducible characters χ_j :

$$\phi_i = \sum n_{ij} \chi_j$$

The numbers n_{ij} form an $a \times b$ matrix which defines a \mathbb{Q} linear map:

$$(n_{ij}) : \mathbb{Q}^b \rightarrow \mathbb{Q}^a$$

Since $a < b$ this linear map has a kernel, i.e., there are rational numbers c_j not all zero so that

$$\sum_j n_{ij} c_j = 0 \quad \forall i$$

Multiplying by the denominators, we may assume the numbers c_j are integers. This gives a nonzero virtual character

$$\sum c_j \chi_j = \chi_V - \chi_W$$

which is orthogonal to all the ϕ_i and therefore all $\phi \in L$:

$$\langle \phi_i, \chi_W - \chi_{W'} \rangle = \left\langle \phi_i, \sum c_j \chi_j \right\rangle = \sum n_{ij} c_j = 0$$

But L contains all induced characters:

$$\phi = \text{Ind}_H^G V$$

for all $H \in \mathcal{X}$ and all representations V of H . So, by Frobenius reciprocity, we have:

$$\langle \phi, \chi_W - \chi_{W'} \rangle = \langle \text{Ind}_H^G V, \chi_W - \chi_{W'} \rangle_G = \langle V, \text{Res}_H^G(\chi_W - \chi_{W'}) \rangle_H = 0$$

Since this is true for all representations V of H , we must have

$$\text{Res}_H^G(\chi_W - \chi_{W'}) = 0$$

for all $H \in \mathcal{X}$. This in turn implies that

$$\chi_W(\sigma) = \chi_{W'}(\sigma)$$

for all $\sigma \in H$.

But, for any $\sigma \in G$ there is an $H \in \mathcal{X}$ which contains a conjugate of σ . But then

$$\chi_W(\sigma) = \chi_{W'}(\sigma)$$

So, the virtual character $\chi_W - \chi_{W'}$ must be zero, which is a contradiction. This proves the theorem. \square