121B: ALGEBRAIC TOPOLOGY

1. Introduction and examples

In the first week I am doing the definition of simplicial cohomology and some examples.

1.1. Δ-complex.

Here is a formal definition of a Δ-complex.

Definition 1.1. A Δ-*complex* X is a sequence of sets $X_0, X_1, \cdots$ (X_n is the set of n-simplicies of X) and mappings $\partial_i : X_n \to X_{n-1}$ for $i = 0, \cdots, n$ ($\partial_i(\sigma)$ is the ith face of σ given by deleting the ith vertex) so that:

$$\partial_i \partial_j = \partial_{j-1} \partial_i$$

if $i < j$. (If you delete the ith vertex then the jth vertex becomes the $j-1$st vertex.)

Problem: Find a formula for the ith vertex of $\sigma \in X_n$.

Answer: ________

A Δ-complex X is an *(ordered) simplicial complex* if

(1) Every n-simplex $\sigma \in X_n$ has $n+1$ distinct vertices.
(2) $\sigma \in X_n$ is uniquely determined by its set of vertices.

Under these conditions we can write

$$\sigma = [v_0, v_1, \cdots, v_n]$$

The ith boundary map is given by

$$\partial_i [v_0, v_1, \cdots, v_n] = [v_0, v_1, \cdots, \widehat{v_i}, \cdots, v_n]$$

Hatcher uses this notation even if the vertices are all equal. This is meant to be an intuitive description of σ and its faces.

1.2. Δ-cohomology.

If G is an additive group and X is a Δ complex then an n-*cochain* on X with coefficients in G is any mapping $\phi : X_n \to G$. These form a group $\Delta^n(X;G)$ by pointwise addition: $(\phi + \psi)(x) = \phi(x) + \psi(x)$. Let $\delta : \Delta^n(X;G) \to \Delta^{n+1}(X;G)$ be given by $\delta\phi(x) = \sum (-1)^i \phi(\partial_i x)$. In Hatcher's notation:

$$(\delta\phi)[v_0, v_1, \cdots, v_n] = \sum_{i=0}^{n} (-1)^i \phi[v_0, \cdots, \widehat{v_i}, \cdots, v_n]$$

Elements in the kernel of $\delta : \Delta^n(X;G) \to \Delta^{n+1}(X;G)$ are called n-*cocycles* on X with coefficients in G and elements in the image are the $n+1$ *coboundaries*.

$Z^n(X;G) = \{\phi \in \Delta^n(X;G) \mid \delta\phi = 0\}$

$B^n(X;G) = \{\phi \in \Delta^n(X;G) \mid \phi = \delta\psi \text{ for some } \psi \in \Delta^{n-1}(X;G)\}$

We have the inclusion of additive groups: $B^n(X;G) \subseteq Z^n(X;G) \subseteq \Delta^n(X;G)$. The *nth cohomology group of X with coefficients in G* is defined to be the quotient:

$$H^n(X;G) := \frac{Z^n(X;G)}{B^n(X;G)}$$

Elements of $H^n(X;G)$ are written

$$[\phi] = \phi + B^n(X;G)$$

1.3. **Cohomology of a sphere.** The problem is to compute the cohomology of the sphere. A Δ-complex model for the $n-1$ sphere is $\partial\Delta^n$. I started by pointing out that the reduced homology of the n-simplex is trivial since Δ^n is contractible.

Theorem 1.2. $\overline{H}^*(\Delta^n) = 0$.

We actually need the algebraic version of this. Namely, the augmented cellular chain complex is contractible.

Lemma 1.3. *An exact sequence of free abelian groups is chain contractible.*

You are supposed to do this as homework. You are given an exact sequence:

$$0 \leftarrow C_{-1} \xleftarrow{\partial} C_0 \xleftarrow{\partial} C_1 \xleftarrow{\partial} C_2 \leftarrow \cdots$$

where all groups C_n are free abelian. You need to find a chain contraction, i.e., a sequence of homomorphisms $h_n : C_n \to C_{n+1}$ so that

$$h\partial + \partial h = id$$

You may use the fact that every subgroup of a free abelian group is free.

Corollary 1.4. *The augmented chain complex of Δ^n is chain contractible.*

I pointed out that the first and last terms of the cellular chain complex of Δ^n are both $\mathbb{Z}$:

$$0 \leftarrow \mathbb{Z} \leftarrow C_0(\Delta^n) \leftarrow C_1(\Delta^n) \leftarrow \cdots \leftarrow \underbrace{C_n(\Delta^n)}_{=\mathbb{Z}} \leftarrow 0$$

To finish we need the following lemma:

Lemma 1.5. *If C_* is a contractible chain complex then the cochain complex $C^*(C_*;G)$ is also contractible for any (additive) group G.*

The *cochain complex* $C^*(C_*;G)$ has groups

$$C^k(C_*;G) := \mathrm{Hom}(C_k, G)$$

connected by a sequence of *coboundary maps*

$$C^k(C_*;G) = \mathrm{Hom}(C_k, G) \xrightarrow{\delta} \mathrm{Hom}(C_{k+1}, G) = C^{k+1}(C_*;G)$$

given by the dual of (precomposition with) the boundary operator: $\delta(\phi) = \partial^*(\phi) = \phi \circ \partial$

$$\begin{array}{ccc} C_{k+1} & & \\ {\scriptstyle\partial}\downarrow & \searrow^{\delta(\phi)} & \\ C_k & \xrightarrow[\phi]{} & G \end{array} \qquad\qquad \begin{array}{rl} & \delta(\phi) = \phi\partial \in C^{k+1}(C_*;G) \\ & \uparrow{\scriptstyle\delta} \\ \phi \in & C^k(C_*;G) \end{array}$$

Proof. The chain contraction for $C^*(C_*;G)$ is given by the dual h^* of the chain contraction h for C_*:

$$h^*\delta + \delta h^* = (\partial h + h\partial)^* = id^* = id$$

This uses the fact that $\mathrm{Hom}(-, G)$ is a contravariant functor. $\square$

Theorem 1.6. *The cohomology of $\partial\Delta^n$ is given by $H^k(\partial\Delta^n; G) = G$ for $k = 0, n-1$ and $H^k(\partial\Delta; G) = 0$ otherwise.*

Proof. We have the following exact sequence:

$$0 \to \mathrm{Hom}(\mathbb{Z}, G) \to \underbrace{C^0(C_*; G) \xrightarrow{\delta_0} C^1(C_*; G) \to \cdots \xrightarrow{\delta_{n-2}} C^{n-1}(C_*; G)}_{*} \to C^n(C_*; G) \to 0$$

The middle terms (*) are the same as the cochain complex $\Delta^*(\partial\Delta^n; G)$. Since the entire sequence is exact, the subsequence (*) is exact except at the beginning and end where

$$H^0(\partial\Delta^n; G) = \ker \delta_0 = \mathrm{Hom}(\mathbb{Z}, G) \cong G$$

and

$$H^{n-1}(\partial\Delta^n; G) = \mathrm{coker}\, \delta_{n-1} = \mathrm{Hom}(C_n(\Delta^n), G) = \mathrm{Hom}(\mathbb{Z}, G) \cong G$$

The exactness at other points gives $H^k(\partial\Delta^n; G) = 0$ for $k \neq 0, n-1$. □

2. Universal coefficient theorem

Hatcher proves the UCT twice: once conceptually and again computationally (the traditional method). I'll just do the first one. Here is the outline.

Lemma 2.1. *Two free chain complexes C_*, D_* are homotopy equivalent iff they have the same (i.e., isomorphic) homologies.*

In HW1 you are doing the easiest case when the homology is trivial.

Lemma 2.2. *Homology commutes with direct sum: $H_k(\bigoplus_\alpha C_*^\alpha) \cong \bigoplus_\alpha H_k(C_*^\alpha)$*

In the case when each homology group $H_k(C_*)$ if finitely generated, it is the direct sum of $\mathbb{Z}$'s and $\mathbb{Z}/n$'s. So, we only need to do two examples:

$$C_* = \qquad \cdots \leftarrow 0 \leftarrow \underbrace{\mathbb{Z}}_{k} \leftarrow 0 \leftarrow \cdots$$

with homology $H_k(C_*) = \mathbb{Z}$ and cohomology $H^k(C_*; G) = G$ and

$$D_* = \qquad \cdots \leftarrow 0 \leftarrow \underbrace{\mathbb{Z}}_{k} \xleftarrow{n} \underbrace{\mathbb{Z}}_{k+1} \leftarrow 0 \leftarrow \cdots$$

with homology $H_k(C_*) = \mathbb{Z}_n$ and cohomology given by the following cochain complex:

$$C^*(D_*; G) = \qquad \cdots \to 0 \to \underbrace{G}_{k} \xrightarrow{n} \underbrace{G}_{k+1} \to 0 \to \cdots$$

Which has cohomology:

$$H^k(D_*; G) = \{g \in G \mid ng = 0\}$$
$$H^{k+1}(D_*; G) = G/nG$$

The first group is naturally isomorphic to $\mathrm{Hom}(\mathbb{Z}_n, G)$. The second group is isomorphic to $\mathbb{Z}_n \otimes G$ but the isomorphism is not natural since $\mathbb{Z}_n \otimes G$ is covariant in both variables.

2.1. **Reduction to the free resolution case.** Lemma 2.2 is obvious is you know what the symbols mean. The direct sum of chain complexes is given by

$$\begin{array}{ccc} (C_* \oplus D_*)_n & = & C_n \oplus D_n \\ \Big\downarrow \partial_n^{\oplus} = & & \partial_n^C \oplus \partial_n^D \Big\downarrow \\ (C_* \oplus D_*)_{n-1} & = & C_{n-1} \oplus D_{n-1} \end{array}$$

Or, more generally, $(\bigoplus C^\alpha)_n = \bigoplus C_n^\alpha$ with boundary map defined coordinatewise:

$$\partial^{\oplus}(\cdots, x_\alpha, x_\beta, \cdots) = (\cdots, \partial x_\alpha, \partial x_\beta, \cdots)$$

The kernel and cokernels of $\partial^{\oplus}$ are also given coordinatewise:

$$\ker \partial^{\oplus} = \ker \partial^C \oplus \ker \partial^D = \oplus Z_n^\alpha$$

$$\operatorname{coker} \partial^{\oplus} = \operatorname{coker} \partial^C \oplus \operatorname{coker} \partial^D = \oplus B_n^\alpha$$

with homology

$$H_n(\bigoplus C_*^\alpha) = \bigoplus Z_n^\alpha / \bigoplus B_n^\alpha \cong \bigoplus H_n(C_*^\alpha)$$

Lemma 2.1 has an obvious part and a nonobvious part. The obvious part, or at least the part that you are already supposed to know, is the following.

(1) ($\Rightarrow$) Homotopy equivalent chain complexes have isomorphic homologies.
(2) Next, $C \simeq D$ implies that the cochain complexes are homotopy equivalent:

$$C^*(C; G) = \mathrm{Hom}(C_*, G) \simeq C^*(D; G) = \mathrm{Hom}(D_*, G)$$

(3) This implies that $H^*(C_*; G) \cong H^*(D_*; G)$.

Therefore, once we prove Lemma 2.1 we will know the following.

Theorem 2.3. *The homology of a free chain complex determines its cohomology with any coefficients.*

We broke the (nontrivial part of) Lemma 2.1 into two parts and proved the first part.

Step 1. Every free chain complex C_* is isomorphic to a direct sum of free chain complexes C_*^n of the following form:

$$C_*^n: \qquad \cdots \leftarrow 0 \leftarrow D_n \xleftarrow{\partial_{n+1}} D_{n+1} \leftarrow D_{n+2} \leftarrow 0 \cdots \tag{2.1}$$

where $D = C_*^n$ is exact except possibly at D_n. If the n-homology is H then we have a long exact sequence:

$$0 \leftarrow H \xleftarrow{\epsilon} D_n \xleftarrow{\partial_{n+1}} D_{n+1} \leftarrow D_{n+2} \leftarrow 0 \cdots \tag{2.2}$$

The sequence (2.1) is called a *free resolution* of H.

Step 2. Any two free resolutions of H are chain homotopy equivalent.

To do step 1, we broke up any free chain complex

$$C_*: \qquad \cdots \leftarrow C_{n-1} \xleftarrow{\partial_n} C_n \xleftarrow{\partial_{n+1}} C_{n+1} \leftarrow \cdots$$

at the point C_n. This uses the following short exact sequence:

$$0 \leftarrow B_{n-1} \xleftarrow{\partial_n} C_n \hookleftarrow Z_n \leftarrow 0$$

The exactness of this sequence follows from the definitions of the symbols. Now use the fact that

Proposition 2.4. *Every subgroup of a free abelian group is free abelian.*

Thus B_{n-1} is free which means that the above short exact sequence splits. I.e., there is a homomorphism $s : B_{n-1} \to C_n$ so that $\partial_n s = id_B$. This makes $C_n = sB_{n-1} \oplus Z_n$. But, the image of $\partial_{n+1} : C_{n+1} \to C_n$ lies in Z_n and $\partial_n : C_n \to C_{n-1}$ is zero on Z_n. Therefore the chain complex C_* is isomorphic to the direct sum of the chain complexes:

$$\cdots \leftarrow C_{n-1} \xleftarrow{\partial_n} sB_{n-1} \leftarrow 0 \leftarrow \cdots$$

$$\cdots \leftarrow 0 \leftarrow Z_n \xleftarrow{\partial_{n+1}} C_{n+1} \leftarrow \cdots$$

The first sequence is exact at degree n and above. The second sequence is zero in degrees $< n$. So, if we do this at every place where the homology of C_* is nonzero, we get a decomposition of C_* as desired.

The traditional method is to do the above construction at every C_n, replacing each C_n with $sB_{n-1} \oplus Z_n$ to get a direct sum of the free chain complexes:

$$\cdots \leftarrow 0 \leftarrow Z_n \xleftarrow{\partial_{n+1}} sB_n \leftarrow 0 \leftarrow \cdots$$

2.2. **Free resolutions and Ext.** It remains to explain the following important theorem.

Theorem 2.5. *Any abelian group has a free resolution and any two free resolutions are chain homotopy equivalent.*

Serge Lang, in his book *Algebra*, said "Take any book on homological algebra, and prove all the theorems without looking at the proofs in that book." Homological algebra has advanced a lot in 40 years. However, it is probably good advice to prove the basic theorems yourself. You just need to know the exact statements of the theorems.

Proof. The existence of a free resolution is easy. Suppose that G is any abelian group. Then there exists a free abelian group F which maps onto G. (You could, e.g., take F to be the free group generated by all the elements of G.) The kernel of the epimorphism $F \to G$ is a subgroup R of F which is also free since all subgroups of free abelian groups are free. Then

$$0 \leftarrow G \leftarrow F \leftarrow R \leftarrow 0 \tag{2.3}$$

is a free resolution of G. This is called a *free presentation* of G. (A free resolution with $C_n = 0$ for $n \geq 2$.)

The following lemma shows the uniqueness of free resolutions up to homotopy: Given any two free resolutions C_*, D_* of the same group G, the identity mapping $G \to G$ is covered by chain maps $f_* : C_* \to D_*$ and $g_* : D_* \to C_*$. The composition $g_* f_* : C_* \to C_*$ must be homotopic to the identity chain map $C_* \to C_*$ by the lemma since both cover the identity map $G \to G$. $\square$

Lemma 2.6. *Suppose that G, H are abelian groups with free resolutions:*

$$0 \leftarrow G \xleftarrow{\epsilon} C_0 \xleftarrow{\partial} C_1 \leftarrow \cdots$$

$$0 \leftarrow H \xleftarrow{\epsilon} D_0 \xleftarrow{\partial'} D_1 \leftarrow \cdots$$

Suppose that $f : G \to H$ is any homomorphism. Then there exist a chain map $f_* : C_* \to D_*$ so that $\epsilon f_0 = f\epsilon : C_0 \to H$. (We say that f_* covers *f.) Furthermore, any two such chain maps are chain homotopy equivalent.*

Proof. Since C_0 is free, the homomorphism $f\epsilon : C_0 \to H$ lifts to a homomorphism $f_0 : C_0 \to D_0$. The homomorphism $f_0\partial : C_1 \to D_0$ has image in $\ker \epsilon = \operatorname{im} \partial'$. So it lifts to a map $f_1 : C_1 \to D_1$, and so on.

If there are two chain maps f_*, g_* covering the same homomorphism $f : G \to H$ then the difference $f_* - g_*$ covers $0 : G \to H$. So, $f_0 - g_0$ has image in $\ker \epsilon = \operatorname{im} \partial'$ and therefore lifts to a homorphism $h_0 : C_0 \to D_1$, etc. The mappings h_n give a chain homotopy $h : f_* \simeq g_*$. I.e.,

$$\partial'_{n+1} h_n + h_{n-1}\partial_n = f_n - g_n$$

□

Now we can define the functor $\operatorname{Ext} = \operatorname{Ext}^1_{\mathbb{Z}}$.

Definition 2.7. If G, H are abelian groups then $\operatorname{Ext}^n_{\mathbb{Z}}(H, G)$ is defined to be the nth cohomology of the cochain complex[1] $\operatorname{Hom}(C_*, G)$ where C_* is any free resolution of H.

By Theorem 2.5 we can choose any C_*. In particular, we can take a free presentation:

$$0 \leftarrow H \leftarrow C_0 \xleftarrow{f} C_1 \leftarrow 0$$

Then $\operatorname{Hom}(C_*, G)$ is the chain complex:

$$0 \to \operatorname{Hom}(C_0, G) \xrightarrow{f^\#} \operatorname{Hom}(C_1, G) \to 0$$

I am using the symbol $f^\#$ to indicate the dual of f given by pre-composition with f because f^* is overused. Since this cochain complex has only two terms, it only has cohomology in degrees $0, 1$. In degree 0, the kernel of $f^\#$ is obviously isomorphic to $\operatorname{Hom}(H, G)$ since any mapping $C_0 \to G$ which is zero on C_1 factors uniquely through the cokernel of $f : C_1 \to C_0$.

Proposition 2.8. $\operatorname{Ext}^n_{\mathbb{Z}}(H, G) = 0$ *for* $n \geq 2$ *and* $\operatorname{Ext}^0_{\mathbb{Z}}(H, G) = \operatorname{Hom}(H, G)$.

This means that the only interesting term is $\operatorname{Ext}^1_{\mathbb{Z}}$ which we call simply Ext

$$\operatorname{Ext}(H, G) := \operatorname{Ext}^1_{\mathbb{Z}}(H, G) = \operatorname{coker} f^\#$$

With the following theorem we can compute $\operatorname{Ext}(H, G)$ for any finitely generated abelian group H.

Theorem 2.9. *For any abelian group G we have:*

1. $\operatorname{Ext}(H_1 \oplus H_2, G) \cong \operatorname{Ext}(H_1, G) \oplus \operatorname{Ext}(H_2, G)$
2. $\operatorname{Ext}(\mathbb{Z}, G) = 0$
3. $\operatorname{Ext}(\mathbb{Z}_n, G) \cong G/nG$

[1] A *cochain complex* is a sequence of abelian groups C^n and homomorphisms $\delta : C^n \to C^{n+1}$ so that $\delta^2 = 0$. The *cohomology* of C^* is the kernel of δ modulo the image of δ.

2.3. **UCT.** Now that we have the definitions, we can state and prove the universal coefficient theorem for cohomology.

Theorem 2.10. *If C_* is a free chain complex and G is an abelian group then we have a natural exact sequence:*

$$0 \to \operatorname{Ext}(H_{n-1}(C_*), G) \to H^n(C_*; G) \to \operatorname{Hom}(H_n(C_*), G) \to 0$$

Furthermore the sequence splits, but not naturally.

Proof. We know that C_* is a direct sum of chain complexes $C_* = \bigoplus_k C_*^{(k)}$ where $C_*^{(k)}$ is a free resolution of $H_k(C_*)$ shifted up in degree by k. So,

$$H^n(C_*; G) \cong \prod_k H^n(C_*^{(k)}; G) = \prod_k \operatorname{Ext}_{\mathbb{Z}}^{n-k}(H_k(C_*), G)$$

Since $\operatorname{Ext}_{\mathbb{Z}}^{n-k} = 0$ except when $k = n, n-1$ we have a finite product (= direct sum):

$$H^n(C_*; G) \cong \operatorname{Ext}_{\mathbb{Z}}^0(H_n(C_*), G) \oplus \operatorname{Ext}_{\mathbb{Z}}^1(H_{n-1}(C_*), G)$$
$$= \operatorname{Hom}(H_n(C_*), G) \oplus \operatorname{Ext}(H_{n-1}(C_*), G)$$

A more detailed description of this isomorphism shows which part is natural. I am skipping this part. The important part is the naturality of the evaluation map

$$ev : H^n(C_*; G) \to \operatorname{Hom}(H_n(C_*), G)$$

This map has the following natural description. Let $f \in \operatorname{Hom}(C_n, G)$ be a cocycle representing a cohomology class $[f] \in H^n(C_*; G)$ and let $z \in C_n$ be a cycle representing the homology class $[z] \in H_n(C_*)$. Then $ev[f]$ is the homomorphism $H_n(C_*) \to G$ which sends $[z]$ to $f(z)$. If $f' = f + \delta g$ and $z' = z + \partial y$ are any other choices for f, z then

$$f'(z') = (f + g\partial)(z + \partial y) = f(z) + f\partial(y) + g\partial z + g\partial^2 y = f(z)$$

since $f\partial = \delta f = 0$, $\partial z = 0$ and $\partial^2 = 0$. So, ev is independent of all choices, i.e., it is well-defined. "Naturality" of ev is obvious if you know what it means. ☐

2.4. **Problems.** **Problem 1**. Find a free presentations of $\mathbb{Z}_n$ and show that

$$\operatorname{Ext}(\mathbb{Z}_n, G) \cong G/nG$$

Conclude that $G/6G \cong G/2G \oplus G/3G$.

Problem 2. Since $\mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$ we get two free presentations of $\mathbb{Z}_6$. Find an explicit chain homotopy equivalence between these free resolutions.

Problem 3. What does it mean that the evaluation map

$$ev : H^n(C_*; G) \to \operatorname{Hom}(H_n(C_*), G)$$

is *natural*?

Homework 2 This is the first problem in HW2. Find a free presentation for the additive group

$$\mathbb{Z}[\tfrac{1}{p}] = \{\tfrac{a}{b} \in \mathbb{Q} \mid a, b \in \mathbb{Z} \text{ and } b \text{ is a power of } p\}$$

(p is prime) and use it to compute $\operatorname{Ext}(\mathbb{Z}[\frac{1}{p}], \mathbb{Z})$.

3. Cohomology of spaces

Using the universal coefficient theorem and our knowledge of homology, we can compute the cohomology of spaces. The general properties of cohomology follow from the corresponding property for homology by duality. The evaluation map is also a consequence of duality:

$$\text{computation of } H_n \quad \Rightarrow_{UCT} \quad \text{computation of } H^n$$

$$\text{properties of } H_n \quad \Rightarrow_{\text{Hom}} \quad \text{properties of } H^n$$

3.1. **Singular cohomology.** This is derived from singular homology. If X is a topological space then recall that $C_*(X)$ is the singular chain complex of X. $C_n(X)$ is the free abelian group generated by the set of all continuous mappings $\sigma : \Delta^n \to X$. These are called *singular n-simplices* in X and $\partial : C_n(X) \to C_{n-1}(X)$ is given by alternating sum of faces.

Definition 3.1. The *singular cochain complex* of a spaces X with coefficients in G is the sequence of groups $C^n(X;G) := \text{Hom}(C_n(X);G)$ with coboundary map $\delta : C^n(X;G) \to C^{n+1}(X;G)$ given by $\delta = \partial^\#$.

$$\delta(\phi)\sigma = \phi(\partial\sigma) = \sum_{i=0}^{n} \phi(\sigma \,|\, [v_0, \cdots, \widehat{v_i}, \cdots, v_n])$$

3.1.1. *UCT and computations.* Since $C_*(X)$ is a free chain complex, the UCT for free chain complexes (Theorem 2.10) implies the UCT for spaces.

Theorem 3.2 (UCT for spaces)**.** *For any space X and any additive group G we have a natural exact sequence:*

$$0 \to \text{Ext}(H_{n-1}(X), G) \to H^n(X;G) \xrightarrow{h} \text{Hom}(H_n(X), G) \to 0$$

where h is the evaluation map. Furthermore this sequence splits, but not naturally.

The evaluation map h is given by $h[\phi]([z]) = \phi(z)$. In the special case when $G = F$ is a field (e.g., $F = \mathbb{Z}_p, \mathbb{Q}, \mathbb{R}, \mathbb{C}$) there is a simpler theorem:

Theorem 3.3 (UCT over a field)**.** *The evaluation map gives a natural isomorphism:*

$$h : H^n(X;F) \xrightarrow{\approx} \text{Hom}(H_n(X;F), F)$$

This does not follow from the UCT for free modules (= vector spaces) over the field F instead of free abelian groups and the fact that $\text{Ext}^n_F(V,W) = 0$ for all $n > 0$.

Example 3.4. Take the projective plane $X = P^2 = \mathbb{R}P^2$. We know that $H_0(P^2) = \mathbb{Z}, H^1(P^2) = \mathbb{Z}_2$ and $H^n(P^2) = 0$ for $n \geq 2$. So, the UCT tells us that

(0) $H^0(P^2;G) \cong \text{Hom}(H_0(P^2), G) = G$.
(1) $H^1(P^2;G) \cong \text{Hom}(H_1(P^2), G) \oplus \text{Ext}(H_0(P^2); G) \cong \text{Hom}(\mathbb{Z}_2, G)$. This is the group of elements of G of order 2 (or 1).
(2) $H^2(P^2;G) \cong \text{Ext}(H_1(P^2), G) = \text{Ext}(\mathbb{Z}_2, G) = G/2G$.

In the special case when $G = \mathbb{Z}_2$ this gives $H^0 = H^1 = H^2 = \mathbb{Z}_2$ and $H^n = 0$ for $n > 2$. This agrees with the second UCT since $H_n(P^2;\mathbb{Z}_2) = \mathbb{Z}_2$ for $n = 0, 1, 2$.

Problem Describe $H^0(X;G)$ for any X, G.

3.1.2. *Properties of cohomology.* If X is a Δ-complex or a CW-complex, the cellular chain complex $\Delta_*(X)$ has the same homology as the singular chain complex $C_*(X)$. Therefore, they are homotopy equivalent:

$$\Delta_*(X) \simeq C_*(X).$$

So, the cochain complexes are homotopy equivalent:

$$\Delta^*(X; G) = \text{Hom}(\Delta_*(X), G) \simeq \text{Hom}(C_*(X); G) = C^*(X; G)$$

This proves the following theorem.

Theorem 3.5. *If X is a Δ-complex or a CW-complex, its cellular cohomology $H^*_\Delta(X; G)$ is isomorphic to its singular cohomology $H^*(X; G)$.*

Example 3.6. The cellular chain complex of P^2 using the CW decomposition $P^2 = * \cup e^1 \cup e^2$ is given by:

$$0 \leftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \leftarrow 0.$$

The cellular cochain complex is $\Delta^*(P^2; G) = \text{Hom}(\Delta_*(P^2), G)$:

$$0 \to G \xrightarrow{0} G \xrightarrow{2} G \to 0.$$

If $f : X \to Y$ is a continuous mapping then we have a chain map $f_\# : C_*(X) \to C_*(Y)$ given by left composition with f: $f_\#(\sigma) = f \circ \sigma : \Delta \to Y$. This induces a cochain map:

$$f^\# : C^*(Y; G) \to C^*(X; G)$$

given by right composition with f: $f^\#(\phi) = \phi \circ f$. So,

$$f^\#(\phi)(\sigma) = \phi(f \circ \sigma) = \phi(f_\#(\sigma))$$

The cochain map $f^\#$ induces a map (homomorphism) in cohomology:

$$f^* : H^*(Y; G) \to H^*(X; G)$$

The key point is that it goes the wrong way (Y to X). So, cohomology is a *contravariant* functor.

Theorem 3.7. *If $f, g : X \to Y$ are homotopic then they induce the same map in cohomology:*

$$f^* = g^* : H^*(Y; G) \to H^*(X; G)$$

Proof. We know that $f_\#, g_\# : C_*(X) \to C_*(Y)$ are chain homotopic. Let h be a chain homotopy. Thus

$$\partial h + h\partial = g_\# - f_\#$$

Applying the functor $\text{Hom}(-, G)$ to everything we get:

$$h^\# \delta + \delta h^\# = g^\# - f^\#$$

Therefore, $h^\#$ gives a homotopy $f^\# \simeq g^\#$. So, $f^\#, g^\#$ induce the same map in cohomology. In case these symbols don't mean anything to you, this follows from the previous equation by duality:

$$(h^\# \delta + \delta h^\#)(\phi) = \phi \circ \partial \circ h + \phi \circ h \circ \partial = \phi(\partial h + h\partial)$$
$$= \phi(g_\# - f_\#) = \phi \circ g + \phi \circ f = (g^\# - f^\#)(\phi)$$
$$(h^\# \delta)(\phi) = \phi \circ \partial \circ h : C_n(X) \xrightarrow{h} C_{n+1}(Y) \xrightarrow{\partial} C_n(Y) \xrightarrow{\phi} G$$

□

HW2(b) Do problem 11 on p. 205.

3.2. **Relative cohomology and long exact sequences.** The definition of the relative cohomology is dual to relative homology and the long exact cohomology sequence of a pair is given by the same algebraic lemma, namely that a short exact sequence of cochain complexes gives a long exact sequence in cohomology. (Some authors use the formal trick: $C^n = C_{-n}$ to convert a cochain complex into a chain complex.)

If A is a subspace of X then we have the short exact sequence of free chain complexes:

$$0 \to C_*(A) \xrightarrow{j_\#} C_*(X) \to C_*(X, A) \to 0$$

It is important that $C_*(X, A)$ is a free chain complex: $C_n(X, A)$ is the free abelian group generated by singular n-simplices $\sigma : \Delta^n \to X$ whose images do not lie in A.

When we hom this into a group G we get a short exact sequence of cochain complexes:

$$0 \to C^*(X, A; G) \to C^*(X; G) \xrightarrow{j^\#} C^*(A; G) \to 0 \tag{3.1}$$

The *relative cohomology groups* $H^*(X, A; G)$ are the cohomology groups of the cochain complex $C^*(X, A; G)$.

3.2.1. *Interpretation of relative cohomology.* Cohomology is nicer than homology in many ways. For example $C_n(X, A) = C_n(X)/C_n(A)$ is a quotient group and elements of $C_n(X, A)$ are cosets of $C_n(A)$. But $C^n(X, A; G)$ is naturally a subset of $C^n(X; G)$. (It is the kernel of the restriction map $j^\#$.) Therefore, we have the following interpretation of relative cochains, cocycles and coboundaries.

(1) A *relative cochain* $\phi \in C^n(X, A; G)$ is a function which assigns to each singular simplex $\sigma : \Delta^n \to X$ an element $\phi(\sigma) \in G$ so that $\phi(\sigma) = 0$ if $\sigma(\Delta^n) \subset A$.
(2) ϕ is a *relative cocycle* if $\delta\phi = 0$. I.e.,

$$Z^n(X, A; G) = C^n(X, A; G) \cap Z^n(X; G)$$

(3) ϕ is a *relative coboundary* if $\phi = \delta\psi$ for some $\psi \in C^{n-1}(X, A; G)$.

3.2.2. *Computation of relative cohomology.* Since $C_*(X, A)$ is a free chain complex, its cohomology can be computed using the UCT.

Theorem 3.8. *There is a natural short exact sequence which splits unnaturally:*

$$0 \to \operatorname{Ext}(H_{n-1}(X, A), G) \to H^n(X, A; G) \xrightarrow{h} \operatorname{Hom}(H_n(X, A), G) \to 0$$

where h is the evaluation map:

$$h([\phi])(z + C_n(A) + B_n(X)) = \phi(z)$$

$\phi(z) \in G$ is well defined: $\phi = 0$ on $C_n(A)$ by definition of a relative cochain and $\phi = 0$ on $B_n(X)$ since $\delta\phi = 0$.

For example, if X is obtained from a space A by attaching r number of n-cells $e_1^n, \cdots, e_k^n$ along maps $\eta_i : S^{n-1} \to A$ then $H_n(X, A) = \mathbb{Z}^r$ and $H_i(X, A) = 0$ for $i \neq n$. So,

$$H^n(X, A; G) \cong \operatorname{Hom}(\mathbb{Z}^r, G) = G^r$$

3.2.3. *Long exact sequences.* The short exact sequence of cochain complexes (3.1) gives a long exact sequence in cohomology, just as in the homology case:

$$\cdots \to H^n(X, A; G) \to H^n(X; G) \xrightarrow{j^*} H^n(A; G) \xrightarrow{\delta} H^{n+1}(X, A; G) \to \cdots$$

The mapping δ is given by diagram chasing, just like before.

3.2.4. *Problems.*

(1) What is the definition of the connecting homomorphism δ in the long exact sequence?
(2) Find an example of the previous example where $\delta : H^{n-1}(A; G) \to H^n(X, A; G)$ is nonzero.
(3) Show that the composition
$$\operatorname{Ext}(H_{n-2}(A), G) \to H^{n-1}(A; G) \xrightarrow{\delta} H^n(X, A; G)$$
is zero when $X = A \cup e_1^n \cup \cdots \cup e_r^n$.

We did (1) and (3) in class. Do (2) for homework.

4. Cup product

If the coefficient group is (the underlying additive group of) a ring R. Thus we have a multiplication map $\mu : R \otimes R \to R$ which is associative with unit. Then the cohomology groups $H^n(X;R)$ form a graded ring $\bigoplus H^n(X;R)$. It is very important that this ring structure is *defined at the cochain level.* I.e., $\bigoplus C^n(X;R)$ has an associative ring structure. The multiplication is call *cup product.*

4.1. **Cup product at the cochain level.**

Here is my favorite formula for the cup product:

$$\phi \cup \psi = \mu \circ (\phi \otimes \psi) \circ \Delta$$

Here $\phi \in C^k(X;R), \psi \in C^\ell(X;R)$ and $\phi \cup \psi \in C^{k+\ell}(X;R) = \mathrm{Hom}(C_{k+\ell}(X), R)$ is given by the composition

$$C_{k+\ell}(X) \xrightarrow{\Delta_{k,\ell}} C_k(X) \otimes C_\ell(X) \xrightarrow{\phi\otimes\psi} R \otimes R \xrightarrow{\mu} R$$

4.1.1. *The diagonal map* Δ. The diagonal map is given by "front face" and "back face."

$$\Delta_{k,\ell}(\sigma) = f_k\sigma \otimes b_\ell\sigma$$

where $f_k\sigma$ is the *front* k*-face* of σ given by

$$f_k\sigma = \sigma|[v_0, v_1, \cdots, v_k]$$

and $b_\ell\sigma$ is the *back* ℓ*-face* of σ given by

$$b_\ell\sigma = \sigma|[v_k, \cdots, v_{k+\ell}]$$

Thus, the cup product is given by:

$$(\phi \cup \psi)(\sigma) = \mu(\phi \otimes \psi)\Delta(\sigma) = \mu(\phi \otimes \psi)(f_k\sigma \otimes b_\ell\sigma) = \phi(f_k\sigma)\psi(b_\ell\sigma)$$

$$= \phi(\sigma|[v_0, \cdots, v_k])\psi(\sigma|[v_k, \cdots, v_{k+\ell}])$$

Lemma 4.1. *We have the following identities of operators on* $C_{i+j+k}(X)$*.*

(1) $f_i f_{i+j} = f_i$*: the front* i*-face of the front* $i+j$*-face is the front* i*-face.*
(2) $b_k b_{j+k} = b_k$*: the back face of the back face is the back face.*
(3) $b_j f_{i+j} = f_j b_{j+k}$*: this is the* middle face $m_j\sigma = \sigma|[v_i, \cdots, v_{i+j}]$.

In terms of the diagonal maps defined on $C_{i+j+k}(X)$ these three formulas give one formula:

$$(\Delta_{ij} \otimes id) \circ \Delta_{i+j,k} = (id \otimes \Delta_{jk}) \circ \Delta_{i,j+k}$$

When we apply these operators to σ we get:

$$f_i(f_{i+j}\sigma) \otimes b_j(f_{i+j}\sigma) \otimes b_k\sigma = f_i\sigma \otimes f_j(b_{j+k}\sigma) \otimes b_k(b_{j+k}\sigma)$$

When we sum over all i, j, k we get:

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$$

i.e., Δ is *coassociative.*

4.1.2. *Properties of cup product.* The coassociativity of Δ (and the associativity of μ) imply that cup product is associative at the cochain level.

Theorem 4.2. *If* $\alpha \in C^i(X;R), \beta \in C^j(X;R), \gamma \in C^k(X;R)$ *then*

$$(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma)$$

Proof. Associativity of

$$(\alpha \cup \beta) \cup \gamma = \mu((\alpha \cup \beta) \otimes \gamma)\Delta = \mu(\mu \otimes id)(\alpha \otimes \beta \otimes \gamma)(\Delta \otimes id)\Delta$$
$$\alpha \cup (\beta \cup \gamma) = \mu(\alpha \otimes (\beta \cup \gamma))\Delta = \mu(id \otimes \mu)(\alpha \otimes \beta \otimes \gamma)(id \otimes \Delta)\Delta$$

But $\mu(\mu \otimes id) = \mu(id \otimes \mu)$ by associativity of μ and $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$ by coassociativity of Δ □

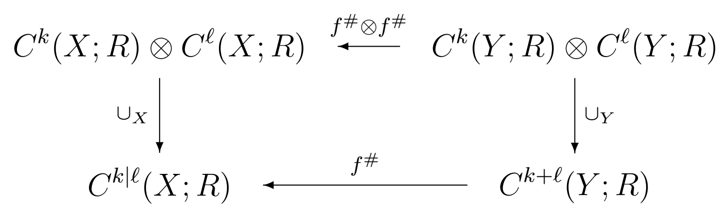
Problem Show that the augmentation map $\epsilon : C_0(X) \to R$ (given by composing the usual augmentation map $\epsilon : C_0(X) \to \mathbb{Z}$ with the unique ring homomorphism $\mathbb{Z} \to R$) is the unit for cup product, i.e.

$$\epsilon \cup \phi = \phi = \phi \cup \epsilon$$

Problem Show that cup product is natural, i.e., given any continuous mapping $f : X \to Y$ and $\phi \in C^k(Y;R), \psi \in C^\ell(Y;R)$ we have:

$$f^\#(\phi \cup \psi) = f^\#(\phi) \cup f^\#(\psi)$$

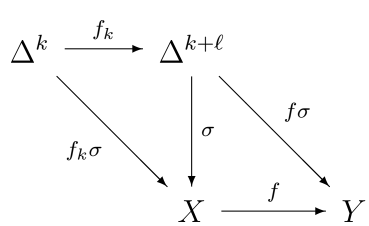
In other words, the following diagram commutes.



By looking at this diagram and the definition of $\cup$ we realized that we need to show:

$$\phi(ff_k\sigma)\psi(fb_\ell\sigma) = \phi(f_k(f\sigma))\psi(b_\ell(f\sigma))$$

I.e., we need to know that $ff_k\sigma = f_k(f\sigma)$ and $fb_\ell\sigma = b_\ell(f\sigma)$. To show the first one we looked at another diagram:



We realized that the front k-face of a singular $k+\ell$ simplex $\sigma : \Delta^{k+\ell} \to X$ is actually a composition: $f_k\sigma = \sigma \circ f_k$ where $f_k : \Delta^k \to \Delta^{k+\ell}$ is the inclusion of the front k-face. Therefore:

$$f(f_k\sigma) = f \circ (\sigma \circ f_k) = (f \circ \sigma) \circ f_k = f_k(f\sigma).$$

4.1.3. *Derivation.* The main property of cup product is that δ is a graded derivation:

Theorem 4.3.
$$\delta(\phi \cup \psi) = \delta\phi \cup \psi + (-1)^k \phi \cup \delta\psi$$
if $\phi \in C^k(X; R)$.

Proof. Apply both sides to any $\sigma : \Delta^{k+\ell+1} \to X$.

$$\delta(\phi \cup \psi)\sigma = (\phi \cup \psi)\partial\sigma = \sum_{i=0}^{k+\ell+1} (-1)^i (\phi \cup \psi)\partial_i\sigma = \sum_{i=0}^{k+\ell+1} (-1)^i \phi(f_k\partial_i\sigma)\psi(b_\ell\partial_i\sigma) \tag{4.1}$$

This expression has $k + \ell + 2$ terms.

$$(\delta\phi \cup \psi)\sigma = \delta\phi(f_{k+1}\sigma)\psi(b_\ell\sigma) = \sum_{i=0}^{k+1} (-1)^i \phi(\partial_i f_{k+1}\sigma)\psi(b_\ell\sigma) \tag{4.2}$$

But $\partial_i f_{k+1}\sigma = f_k\partial_i\sigma$ and $b_\ell\sigma = \sigma|[v_{k+1}, \cdots, v_{k+\ell+1}] = b_\ell\partial_i\sigma$ for $i \leq k$. So, the terms with $i \leq k$ in (4.2) are equal to the first $k + 1$ terms in (4.1).

$$(-1)^k(\phi \cup \delta\psi)\sigma = (-1)^k\phi(f_k\sigma)\delta\psi(b_{\ell+1}\sigma) = \sum_{j=0}^{\ell+1} (-1)^{k+j}\phi(f_k\sigma)\psi(\partial_j b_{\ell+1}\sigma) \tag{4.3}$$

Here $\partial_j b_{\ell+1}\sigma = b_\ell\partial_{j+k}\sigma$ and, for $j \geq 1$, $f_k\sigma = f_k\partial_{j+k}\sigma$. So, the terms with $j \geq 1$ in (4.3) are equal to the last $\ell + 1$ terms in (4.1).

Finally, the last term of (4.2) cancels the first terms of (4.3). They are
$$\phi(f_k\sigma)\psi(b_\ell\sigma)$$
with opposite signs. □

4.2. **Cup product in cohomology.** Theorem 4.3 implies that cup product induces a multiplication in cohomology.

(1) If ϕ, ψ are both cocycles then $\phi \cup \psi$ is a cocycle.
(2) If ϕ is a coboundary $\phi = \delta\alpha$ and ψ is a cocycle then
$$\delta(\alpha \cup \psi) = \delta\alpha \cup \psi \pm \alpha \cup \delta\psi = \phi \cup \psi$$
So, $\phi \cup \psi$ is a coboundary.
(3) Similarly, if ϕ is a cocycle and ψ is a coboundary, then $\phi \cup \psi$ is a coboundary.

Corollary 4.4. *Cup product induces a multiplication in cohomology:*
$$\cup : H^k(X; R) \otimes H^\ell(X; R) \to H^{k+\ell}(X; R)$$
by $[\phi] \cup [\psi] = [\phi \cup \psi]$. *Furthermore, this multiplication is associative with unit. So,*
$$\bigoplus H^n(X; R)$$
is a graded ring for any associative ring R.

The word *graded ring* means that all elements are sums of homogeneous elements and the product of a homogeneous element of degree k with a homogeneous element of degree ℓ is a homogeneous element of degree $k + \ell$. E.g., polynomial rings are graded but power series rings are not.

4.2.1. *Note on tensor product.* The coefficient ring R will be suppressed from the notation. Also, we will from now on, assume that R is commutative.

I actually only defined cup product on the Cartesian product

$$C^k(X) \times C^\ell(X) \xrightarrow{\cup} C^{k+\ell}(X)$$

However, the cup product is *biadditive*, i.e., $(\phi_1+\phi_2)\cup\psi = \phi_1\cup\psi+\phi_2\cup\psi$ and $\phi\cup(\psi_1+\psi_2) = \phi\cup\psi_1+\phi\cup\psi_2$. So, it induces a map on the tensor product

$$C^k(X) \otimes C^\ell(X) \xrightarrow{\cup} C^{k+\ell}(X)$$

It is also R-bilinear in the sense that it also satisfies $(r\phi)\cup\psi = r(\phi\cup\psi) = \phi\cup(r\psi)$. So, we get:

$$C^k(X) \otimes_R C^\ell(X) \xrightarrow{\cup} C^{k+\ell}(X)$$

4.2.2. *Cup product on relative cohomology.* If $A \subseteq B \subseteq X$ then the cup product gives a map:

$$C^k(X,A) \otimes_R C^\ell(X,B) \xrightarrow{\cup} C^{k+\ell}(X,B)$$

We analyzed why this is true.

1. If $\phi \in C^k(X,A)$ then $\phi(\sigma) = 0$ whenever $\sigma(\Delta^k) \subseteq A$.
2. If $\psi \in C^\ell(X,B)$ then $\psi(\sigma) = 0$ whenever $\sigma(\Delta^\ell) \subseteq B$.
3. So, $(\phi\cup\psi)\sigma = \phi(f_k\sigma)\psi(b_\ell\sigma) = 0$ if $\sigma(\Delta^\ell) \subseteq B$.

It would also work to put $C^{k+\ell}(X,A)$ instead of $C^{k+\ell}(X,B)$. I said that it was usually "better" to use B instead of A. In cohomology we get:

$$H^k(X,A) \otimes_R H^\ell(X,B) \xrightarrow{\cup} H^{k+\ell}(X,B)$$

4.2.3. *Cross product.*

Definition 4.5. For a Cartesian product $X \times Y$ the *cross product*

$$\times : H^k(X;R) \otimes_R H^\ell(Y;R) \to H^{k+\ell}(X\times Y;R)$$

is defined by

$$\alpha \times \beta = p_X^*(\alpha) \cup p_Y^*(\beta)$$

where p_X^*, p_Y^* are the maps in cohomology induced by the projection maps $p_X : X\times Y \to X$ and $p_Y : X \times Y \to Y$.

The cross product is defined at the cochain level by

$$\phi \times \psi = p_X^\#(\phi) \cup p_Y^\#(\psi)$$

5. Künneth formula

The Künneth formula gives the homology and cohomology of a product of spaces. In my approach to this I used universal objects which are called "acyclic models" and the "Yoneda element." I used the algebraic part of the Künneth formula to prove the geometric part.

5.1. **Logic.** To dispel the appearance of a circular argument, I explained the logic of the proof. The proof is divided into two parts which are independent. The first part is a geometric theorem called the *Eilenberg-Zilber Theorem.*

Theorem 5.1 (Thm A: Eilenberg-Zilber)**.** *The singular chain complex of a product space $X \times Y$ is naturally homotopy equivalent to the tensor product of the singular chain complexes of the factors:*

$$C_*(X \times Y) \simeq C_*(X) \otimes C_*(Y)$$

The second part (Theorem B) computes the homology and cohomology of a tensor product of free chain complexes. This is done in two parts B1, B2. We only need the first part to prove part A (Thm B1 $\Rightarrow$ Thm A).

Theorem 5.2 (Thm B1)**.** *Suppose that C_*, D_* are free chain complexes so that the homology of D_* is free abelian. Then the homology of $C_* \otimes C_*$ is isomorphic to the tensor product of the homologies of C_*, D_*:*

$$H_n(C_* \otimes D_*) \cong \bigoplus_{k+\ell=n} H_k(C_*) \otimes H_\ell(D_*)$$

We will discuss the general formula (Theorem B2) later.

5.2. **Tensor product of chain complexes.** We went over the definition of the tensor product of chain complexes and proved Theorem B1.

Definition 5.3. If C_*, D_* are chain complexes with differentials ∂^C, ∂^D then the chain complex $C_* \otimes D_*$ is defined as follows. The groups in the chain complex are:

$$(C_* \otimes D_*)_n := \bigoplus_{k+\ell=n}$$

The boundary map $\partial^\otimes : C_k \otimes D_\ell \to (C_* \otimes D_*)_{k+\ell-1}$ is given by

$$\partial^\otimes(x \otimes y) = \partial^C x \otimes y + (-1)^k x \otimes \partial^D y$$

The sign $(-1)^k$ insures that $(\partial^\otimes)^2 = 0$.

Lemma 5.4. *If $C_* \simeq C'_*$ and $D_* \simeq D'_*$ then $C_* \otimes D_* \simeq C'_* \otimes D'_*$.*

Proof of Theorem B1. If $H_*(D_*)$ is free abelian then $D_* \simeq H_*(D_*)$ are homotopic free chain complexes (the boundary map being zero in $H_*(D_*)$). Therefore, by the lemma above, $C_* \otimes D_* \simeq C_* \otimes H_*(D_*)$ with homology

$$H_n(C_* \otimes D_*) \cong H_n(C_* \otimes H_*(D_*)) \cong \bigoplus_{k+\ell=n} H_k(C_*; H_\ell(D_*)) \cong \bigoplus_{k+\ell=n} H_k(C_*) \otimes H_\ell(D_*)$$

□

5.3. **Skew-commutativity of cup product.** The first theorem we proved using Theorem B1 and the universal-Yoneda element argument was the skew commutativity of cup product.

Theorem 5.5. *If R is commutative and $\alpha \in H^k(X;R), \beta \in H^\ell(X;R)$ then*

$$\alpha \cup \beta = (-1)^{k\ell} \beta \cup \alpha$$

Proof. I made the definition:

$$\alpha \cup' \beta = (-1)^{k\ell} \beta \cup \alpha$$

We want to show that $\cup' = \cup$. To show this I went to the cocycles. Suppose that ϕ, ψ are cocycles representing α, β and $\sigma : \Delta^{k+\ell} \to X$. Then

$$(\phi \cup \psi)(\sigma) = \phi(f_k \sigma)\psi(b_\ell \sigma)$$

$$(\phi \cup' \psi)(\sigma) = (-1)^{k\ell}(\psi \cup \phi)(\sigma) = (-1)^{k\ell}\phi(b_k\sigma)\psi(f_\ell\sigma)$$

Next, I eliminate all three variables (ϕ, ψ, σ) by using my favorite formula:

$$\phi \cup \psi = \mu(\phi \otimes \psi)\Delta_{k.\ell}$$

$$\phi \cup' \psi = \mu(\phi \otimes \psi)\Delta'_{k.\ell}$$

where $\Delta_{k,\ell}\sigma = f_k\sigma \otimes b_\ell\sigma$ and

$$\Delta'_{k,\ell}\sigma = (-1)^{k\ell} b_k\sigma \otimes f_\ell\sigma$$

So, it suffices to show that $\Delta \simeq \Delta'$. This follows from the following two lemmas.

Lemma 5.6. *Δ and Δ' are natural augmented chain maps*

$$C_*(X) \to C_*(X) \otimes C_*(X)$$

The fact that they are natural and augmented is obvious. (They are augmented since $\epsilon = (\epsilon \otimes \epsilon)\Delta_{0,0} = (\epsilon \otimes \epsilon)\Delta'_{0,0}$.) A calculation shows they are chain maps.

Lemma 5.7. *Any two natural augmented chain maps $C_*(X) \to C_*(X) \otimes C_*(X)$ are naturally chain homotopic.*

Suppose that $f, g : C_*(X) \to C_*(X) \otimes C_*(X)$ are two natural augmented chain maps. Then, the homotopy that we want is a sequence of maps

$$h_n^X : C_n(X) \to (C_*(X) \otimes C_*(X))_{n+1}$$

satisfying two condition

(1) $\partial^\otimes h_n + h_{n-1}\partial = g_n - f_n$
(2) h_n^X is natural, i.e., for any map $f : X \to Y$ we get a commuting square.

Since $C_n(X)$ is freely generated by $\sigma : \Delta^n \to X$, it suffices to find an equation for $h_n^X(\sigma)$. This is the point where I used the universal argument. The mapping h_n^X exists in the universal case when $X = \Delta^n$.

$$h^{\Delta^n} : C_*(\Delta^n) \to C_*(\Delta^n) \otimes C_*(\Delta^n)$$

This is because Δ^n is contractible and thus $C_*(\Delta^n)$ and $C_*(\Delta^n) \otimes C_*(\Delta^n)$ are acyclic (by B1 (Theorem 5.2). Thus, they are both free resolutions of $\mathbb{Z}$ and any two free resolutions of the same abelian group are chain homotopy equivalent by augmented chain maps which are unique up to homotopy. Furthermore, the maps $h_m^{\Delta^n}$ in the homotopy are constructed inductively on $m \geq 0$.

The logic is that, by induction on n, the maps h_{n-1}^X exist for all X and are natural. Then, $h_n^{\Delta^n}$ exists. We apply this map to the *Yoneda element*

$$id_{\Delta^n} : \Delta^n \to \Delta^n \quad \in C_n(\Delta^n)$$

Naturality of h_n^X means that we want the following diagram to commute:

$$\begin{array}{ccc} id_{\Delta^n} \in C_n(\Delta^n) & \xrightarrow{h_n^{\Delta^n}} & (C_*(\Delta^n) \otimes C_*(\Delta^n))_{n+1} \\ \downarrow{\sigma_\#} & & \downarrow{(\sigma\otimes\sigma)_\#} \\ \sigma \in C_n(X) & \xrightarrow{h_n^X} & (C_*(X) \otimes C_*(X))_{n+1} \end{array}$$

This forces the following formula which we took as the definition of $f_n^X(\sigma)$:

$$f_n^X(\sigma) := (\sigma \otimes \sigma)_*(f_n^{\Delta^n}(id_{\Delta^n})$$

In other words, $f_n^X(\sigma)$ is determined for all X and all σ by the value of h_n in the universal case on the Yoneda element. $\square$

5.4. **Homotopy invariance.** The second example of acyclic models and the Yoneda element was the homotopy invariance of homology and cohomology.

Theorem 5.8. *Homotopic mappings $f \simeq g : X \to Y$ induce chain homotopic maps on the singular chain complexes:*

$$f_\# \simeq g_\# : C_* \to D_*$$

By an easy argument which I explained before, this theorem implies that homology and cohomology are homotopy invariant:

Corollary 5.9. *Homotopic maps $f \simeq g : X \to Y$ induce the same map in homology and cohomology with arbitrary coefficients. ($f_* = g_*$ and $f^* = g^*$.)*

Thus it suffices to prove Theorem 5.8.

5.4.1. *Eliminate Y.* We want to eliminate X and Y. I.e., instead of proving the theorem for all X and all Y it suffices to prove it for a particular case where it is obvious. The first step is to eliminate Y as an independent variable.

Problem. It suffices to consider the special case when $Y = X \times I$, $f = j_0$ and $g = j_1$ where $j_t : X \to X \times I$ is the mapping $j_t(x) = (x, t)$. Why?

Answer. Suppose that $h : X \times I \to Y$ is a homotopy from f to g. Then $f = h \circ j_0$ and $g = h \circ j_1$. Therefore, $f_\# = h_\# \circ (j_0)_\#$

$$f_\# : C_*(X) \xrightarrow{(j_0)_\#} C_*(X \times I) \xrightarrow{h_\#} C_*(Y)$$

and $g_\# = h_\# \circ (j_1)_\#$. Now suppose that J is a chain homotopy from $(j_0)_\#$ to $(j_1)_\#$. Then $\partial J + J\partial = (j_1)_\# - (j_0)_\#$ and $h_\# J$ is a homotopy from $f_\#$ to $g_\#$. The reason is that $h_\#$ is a chain map:

$$\partial(h_\# J) + (h_\# J)\partial = h_\# \partial J + h_\# J \partial = h_\#(\partial J + J\partial) = h_\#((j_1)_\# - (j_0)_\#) = g_\# - f_\#$$

5.4.2. *Universal example.* We want to construct a chain homotopy between the two chain maps $(j_0)_\#, (j_1)_\# : C_*(X) \to C_*(X \times I)$ for all spaces X. This will consist of homomorphisms

$$h_n^X : C_n(X) \to C_{n+1}(X \times I)$$

for all spaces X and all $n \geq 0$ satisfying the equation:

$$\partial h_n^X + h_{n-1}^X \partial = (j_1)_\# - (j_0)_\# \tag{5.1}$$

First of all, since $C_n(X)$ is freely generated by the singular n simplices $\sigma : \Delta^n \to X$, the homomorphism h_n^X is determined by the elements $h_n^X(\sigma) \in C_{n+1}(X \times I)$ which can be chosen arbitrarily. However, we also need h_n^X to satisfy (5.1) and we want it to be *natural*. In particular we want the following diagram to commute for all $\sigma : \Delta^n \to X$.

$$\begin{array}{ccc} id_{\Delta^n} \in C_n(\Delta^n) & \xrightarrow{h_n^{\Delta^n}} & C_{n+1}(\Delta^n \times I) \\ \downarrow{\scriptstyle \sigma_\#} & & \downarrow{\scriptstyle (\sigma \times id_I)_\#} \\ \sigma \in C_n(X) & \xrightarrow{h_n^X} & C_{n+1}(X \times I) \end{array}$$

Here Δ^n is the *universal object* and $id_{\Delta^n} \in C_n(\Delta^n)$ is the *Yoneda element.*

We assume by induction on n that h_{n-1}^X has been constructed for all X so that it is natural and satisfies (5.1). Since Δ^n and $\Delta^n \times I$ are contractible, the chain complexes $C_*(\Delta^n)$ and $C_*(\Delta^n \times I)$ acyclic. Therefore, $h_n^{\Delta^n}$ exists satisfying (5.1). Choose one. Then h_n^X can be defined for all X by

$$h_n^X(\sigma) := (\sigma \times id_I)_\# h_n^{\Delta^n}(id_{\Delta^n})$$

It is easy to see that this is natural and satisfies (5.1). This completes the proof of Theorem 5.8.

5.5. **Eilenberg-Zilber.** The last example of acyclic models is the Eilenberg-Zilber Theorem:

$$C_*(X) \otimes C_*(Y) \simeq C_*(X \times Y)$$

We need to construct natural chain maps

$$f : C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$$

and g going the other way so that the compositions $f \circ g, g \circ f$ are homotopic to the identity. First we construct f.

5.5.1. *Construction of f.* We want to construct homomorphisms

$$f_{k,\ell}^{X,Y} : C_k(X) \otimes C_\ell(Y) \to C_{k+\ell}(X \times Y)$$

for all X, Y and all $k, \ell \geq 0$ satisfying two properties:

(1) f is a chain map: $\partial f = f \partial$.
(2) f is natural.

The group $C_k(X) \otimes C_\ell(Y)$ is freely generated by the elements $\sigma \otimes \tau$ where $\sigma : \Delta^k \to X$ and $\tau : \Delta^\ell \to Y$. The *universal example* is given by $X = \Delta^k$ and $Y = \Delta^\ell$. The *Yoneda element* is

$$id_{\Delta^k} \otimes id_{\Delta^\ell} \in C_k(\Delta^k) \otimes C_\ell(\Delta^\ell)$$

Naturality of f requires that the following diagram commute.

$$\begin{array}{ccccc}
id_{\Delta^k} \otimes id_{\Delta^\ell} & \in C_k(\Delta^k) \otimes C_\ell(\Delta^\ell) & \xrightarrow{f_{k,\ell}^{\Delta^k,\Delta^\ell}} & C_{k+\ell}(\Delta^k \times \Delta^\ell) \\
\downarrow & \downarrow{\scriptstyle \sigma_\# \otimes \tau_\#} & & \downarrow{\scriptstyle (\sigma\times\tau)_\#} \\
\sigma \otimes \tau & \in C_k(X) \otimes C_\ell(Y) & \xrightarrow{f_{k,\ell}^{X,Y}} & C_{k+\ell}(X \times Y)
\end{array}$$

Since $C_*(\Delta^k) \otimes C_*(\Delta^\ell)$ and $C_*(\Delta^k \times \Delta^\ell)$ are acyclic, the chain map f^{Δ^k,Δ^ℓ} exists and is unique up to homotopy. Furthermore the construction of this chain map is by induction on degree. Therefore, we may assume by induction on $\kappa + \ell$ that $f_{k,\ell}^{X,Y}$ exists in all smaller degree sums for all X, Y and we may choose $f_{k,\ell}^{\Delta^k,\Delta^\ell}$ so that $\partial f = f \partial$. Then $f_{k,\ell}^{X,Y}$ can be defined for all X, Y by

$$f_{k,\ell}^{X,Y}(\sigma \otimes \tau) := (\sigma \times \tau)_\# f_{k,\ell}^{\Delta^k,\Delta^\ell}(id_{\Delta^k} \otimes id_{\Delta^\ell})$$

5.5.2. *Construction of g.* Now we want to construct a natural chain map $g : C_*(X \times Y) \to C_*(X) \otimes C_*(Y)$. This is a sequence of homomorphisms

$$g_n : C_n(X \times Y) \to (C_*(X) \otimes C_*(Y))_n$$

satisfying $\partial^{\otimes} g_n = g_{n-1}\partial$. Here the universal example is $X = Y = \Delta^n$ and the Yoneda element is the diagonal map $\Delta : \Delta^n \to \Delta^n \times \Delta^n$. The reason that this is the Yoneda element is the following.

A generator of $C_n(X \times Y)$ is a singular n simplex $\sigma : \Delta^n \to X \times Y$. But this is given by the pair of maps $\sigma_X = p_X \circ \sigma$ and $\sigma_Y = p_Y \circ \sigma$ since

$$\sigma(t) = (\sigma_X(t), \sigma_Y(t))$$

The pair of maps σ_X, σ_Y sends the diagonal Δ to $\sigma = (\sigma_X, \sigma_Y)$.

$$\begin{array}{ccccc}
\Delta & \qquad & \in C_n(\Delta^n \times \Delta^n) & \xrightarrow{g_n^{\Delta^n,\Delta^n}} & (C_*(\Delta^n) \otimes C_*(\Delta^n))_n \\
\downarrow & & \downarrow{\scriptstyle (\sigma_X \times \sigma_Y)_\#} & & \downarrow{\scriptstyle (\sigma_X)_\# \otimes (\sigma_Y)_\#} \\
\sigma = (\sigma_X, \sigma_Y) & & \in C_n(X \times Y) & \xrightarrow{g_n^{X,Y}} & (C_*(X) \otimes C_*(Y))_n)
\end{array}$$

As before, we assume by induction on n that a natural map $g_{n-1}^{X,Y}$ has been constructed for all X, Y. Then the chain complexes at the top are acyclic, the map $g_n^{\Delta^n,\Delta^n}$ exists and we may define $g_n^{X,Y}$ by

$$g_n^{X,Y}(\sigma_X, \sigma_Y) := ((\sigma_X)_\# \otimes (\sigma_Y)_\#)\, g_n^{\Delta^n,\Delta^n}(\Delta)$$

5.5.3. *The homotopies.* What I didn't have time to do in class was the homotopies. I'll show that $fg : C_*(X \times Y) \to C_*(X \times Y)$ is homotopic to the identity map. The other way ($gf \simeq id$) is similar. The precise statement is the following.

Lemma 5.10. *Any two natural chain maps $C_*(X \times Y) \to C_*(X \times Y)$ are homotopic.*

Proof. We want to construct a natural chain homotopy

$$h_n^{X,Y} : C_n(X \times Y) \to C_{n+1}(X \times Y)$$

h_{n-1} exists for all X, Y by induction and h_n exists for the universal example $X = \Delta^n, Y = \Delta^n$. The value of $h_n^{\Delta^n,\Delta^n}$ on the Yoneda element Δ determines $h_n^{X,Y}$:

$$h_n^{X,Y}(\sigma_X, \sigma_Y) := (\sigma_X \times \sigma_Y)_\# h_n^{\Delta^n,\Delta^n}(\Delta).$$

□

5.6. **The Yoneda element in general.** The Yoneda element is, by definition, always equal to the identity map of the universal object (or it correspond to the identity map). (What about $\Delta \in C_n(\Delta^n \times \Delta^n)$?) The theorem that we saw several examples of is called the *Yoneda Lemma*. It is considered to be trivial. There are two variations of the Yoneda Lemma, one in the category of sets and one in the category of abelian groups.

5.6.1. *Set theoretic Yoneda Lemma.*

Definition 5.11. Suppose the $\mathcal{C}$ is any category and U is any fixed object of $\mathcal{C}$. (U will be the universal object.) For any other object X of $\mathcal{C}$ let $F(X) = Mor_{\mathcal{C}}(U, X)$ be the set of all morphisms from U to X in the category $\mathcal{C}$. Then $id_U \in F(U)$ is the *Yoneda element* for F.

A functor $F' : \mathcal{C} \to Sets$ is called *representable* if there is a natural isomorphism $F'(X) \cong Mor_{\mathcal{C}}(U, X)$. The element of $F'(U)$ corresponding to the identity map on U is called the *Yoneda element* of F'.

There is a similar definition for contravariant functors: $F(X) = Mor_{\mathcal{C}}(X, U)$ is the representable contravariant functor with universal object U and Yoneda element $id_U \in F(U)$.

Theorem 5.12 (Yoneda Lemma)**.** *Suppose that $G : \mathcal{C} \to Sets$ is any functor and F is as above. Then, a natural map $\phi_X : F(X) \to G(X)$ is given uniquely by*

$$\phi_X(\sigma : U \to X) := \sigma_*(x)$$

where $x = \phi_U(id_U)$ is an arbitrary fixed element of $G(U)$.

This theorem is usually stated as follows: There is a bijection between the set of natural transformations $\phi : F \to G$ and the set of elements of $G(U)$.

5.6.2. *Two examples.* Here are two examples in the homotopy category. First, the categories. Let H be the category whose objects are all nice spaces (those homotopy equivalent to Δ-complexes) and whose morphisms are homotopy classes of maps $[f] : X \to Y$. Thus:

$$Mor_H(X, Y) = [X, Y]$$

where $[X, Y]$ is the standard notation for the set of homotopy classes of maps $X \to Y$.

Let H_0 be the category of pointed nice topological spaces (X, x_0) and pointed homotopy classes of maps $[f] : (X, x_0) \to (Y, y_0)$. This means that $f(x_0) = y_0$ and $f \simeq g$ if the homotopy h is also pointed ($h(x_0, t) = y_0$ for all $t \in I$). So,

$$Mor_{H_0}(X, Y) = [X, Y]_0$$

The base point is understood and indicated by the subscript $_0$.

Example 5.13. Take $\mathcal{C} = H_0$ with universal object $U = (S^n, *)$. Then $\pi_n(X, x_0) := [(S^n, *), (X, x_0)]_0$. The Yoneda element is $[id_{S^n}] \in \pi_n(S^n)$. The Yoneda Lemma says, e.g., that the set of all natural maps $\pi_n(X) \to \pi_m(X)$ are in 1-1 corespondence with the element of $G(U) = \pi_m(S^n)$.

Example 5.14. Take $\mathcal{C} = H$ with some universal object U. Then $F(X) = [X, U]$ is a representable contravariant functor with Yoneda element $[id_U] \in F(U)$. If $G(X) = [X, V]$ is another representable functor then the Yoneda Lemma says that the set of natural transformations $F(X) \to G(X)$ is in 1-1 correspondence with the elements of $G(U) = [U, V]$.

An important special case of this example is singular cohomology:

$$H^n(X; G) \cong [X, K(G, n)]$$

The Yoneda Lemma says that the set of all natural maps $H^n(X; G) \to H^m(X; H)$ is in 1-1 correspondence with the elements of $H^m(K(G, n))$.

Here is an example. Cup square is a natural map $H^n(X; R) \to H^{2n}(X; R)$. By the Yoneda Lemma, it corresponds to some element of $H^{2n}(K(R, n); R)$. What is it?

5.6.3. *Algebraic Yoneda Lemma.* The version of the Yoneda Lemma that we have been using is the following.

Lemma 5.15. *Suppose that $F : \mathcal{C} \to Sets$ is a representable functor $F(X) = Mor_{\mathcal{C}}(U, X)$. For each X in $\mathcal{C}$ let $C(X)$ is the free abelian group generated by $F(X)$. Then C is a functor $\mathcal{C} \to Ab$ and, for any other functor $G : \mathcal{C} \to Ab$, there is a 1-1 correspondence between the set of natural transformations $\phi : C \to G$ and the elements of $G(U)$. We say that C is* free *on the universal object U.*

Proof. Any natural map $\phi_X : C(X) \to G(X)$ is given uniquely by

$$\phi_X(f : U \to X) = f_*(x)$$

where $x = \phi_U(id_U)$ is an arbitrary element of $G(U)$. □

The main example is the singular chain complex $C_*(X)$. For each $n \geq 0$, $C_n(X)$ is free on the universal object Δ^n. The Eilenberg-Zilber Theorem is an example of the following general theorem.

Theorem 5.16 (Acyclic Model Theorem)**.** *Suppose that $C_*(X), D_*(X)$ are augmented chain complexes defined naturally for all objects X of some category $\mathcal{C}$. Suppose that $C_n(X), D_n(X)$ are free with universal objects U_n, V_n. If $C_*(U_n), C_*(V_n), D_*(U_n), D_*(V_n)$ are acyclic for all n then $C_*(X) \simeq D_*(X)$.*

This also holds in the case when $C_n(X), D_n(X)$ are direct sums of free functors whose universal objects are acyclic in both C_* and D_*. Universal objects are also called *models*.

The Acyclic Model Theorem follows from the following lemma. This is also the lemma that we actually used several times.

Lemma 5.17. *Suppose that $C_*(X), D_*(X)$ are augmented chain complexes defined naturally for all objects X of some category $\mathcal{C}$. Suppose that $C_n(X)$ are free on models U_n. If $D_*(U_n)$ are acyclic for all n then there is a natural augmented chain map $C_*(X) \to D_*(X)$ and any two such chain maps are naturally chain homotopic.*

Again, this also holds in the case when $C_n(X)$ is a direct sum of free functors on models $U_{n,i}$ as long as $D_*(U_{n,i})$ are all acyclic.