121B: ALGEBRAIC TOPOLOGY

Contents

7. Homotopy theory

We want to consider path spaces, loop spaces, fibrations and cofibrations and homotopy groups.

7.1. Adjoint maps.

We want to show that, under certain conditions, a continuous mapping $f : X \times Y \to Z$ is equivalent to a continuous mapping of the form

$$\widehat{f} : X \to \mathrm{Map}(Y, Z)$$

called the "adjoint" of f. First we need the mapping space.

Definition 7.1. For Y, Z topological spaces, the *mapping space* $\mathrm{Map}(Y, Z)$ is defined to be the set of all continuous mappings $f : Y \to Z$ with the *compact-open topology* defined as follows. For every compact $K \subseteq Y$ and every open $U \subseteq Z$ we have the *subbasic* open set

$$\langle K, U\rangle := \{f \in \mathrm{Map}(Y, Z) \mid f(K) \subseteq U\}$$

The *basic* open sets are all finite intersections of these. The open sets are arbitrary unions of the basic open sets.

Exercise 7.2. Show that if X, Y are metric spaces, a system of basic open nbhs of $f : X \to Y$ can be given by

$$B(K, \epsilon, f) = \{g : X \to Y \mid d(f(x), g(x)) < \epsilon \ \forall x \in K\}$$

where K is a compact subset of X and $\epsilon > 0$. (To show that two systems of basic open sets give the same topology you need to show that any basic open nbh of f of one kind contains a basic open nbh of the other kind.)

Lemma 7.3. *If $f : X \times Y \to Z$ is continuous then its adjoint $\widehat{f} : X \to \mathrm{Map}(Y, Z)$ is continuous.*

To prove this we need another lemma:

Lemma 7.4 (Hot dog lemma)**.** *If $K \subseteq Y$ is compact, $x \in X$ and $x \times K \subseteq U$ then there exists a "hot dog" $x \times K \subseteq V \times W \subseteq U$ where V is an open nbh of x in X and W is an open subset of Y containing K.*

I'll skip this easy proof.

Proof of 7.3. First we need to show that $\widehat{f}(x) : Y \to Z$ is continuous for all $x \in X$. This means that, for every open subset U of Z, we need to find an open nbh of y which maps into U. But $f^{-1}(U)$ is an open nbh of $(x, y) \in X \times Y$ so there is a product open set $(x, y) \in V \times W \subseteq f^{-1}(U)$. But this means $\widehat{f}(x)(W) \subseteq U$. So, $\widehat{f}$ is continuous.

To show that $\widehat{f}$ is continuous we need to take any subbasic open nbh $\langle K, U\rangle$ of $\widehat{f}(x) \in \mathrm{Map}(Y, Z)$ and find a nbh of x which maps into it. But this is the hot dog lemma. $\widehat{f}(x)(K) = f(x \times K) \subseteq U$ implies there is a "hot dog" $V \times W$ containing $x \times K$ which maps into U. But then $\widehat{f}$ maps V into $\langle K, U\rangle$. $\square$

We had a discussion about he properties of locally compact Hausdorff spaces. Let's just take it as a definition: A Hausdorff space X is *locally compact* if for all $x \in X$ and all open nbhs U of x there is another open nbh V of x so that $\overline{V}$ is compact and contained in U.

Theorem 7.5. *Suppose that Y is locally compact Hausdorff. Then a mapping $f : X \times Y \to Z$ is continuous if and only if its adjoint $\widehat{f} : X \to \mathrm{Map}(Y, Z)$ is continuous.*

Proof. The lemma shows ($\Rightarrow$). So suppose that $\widehat{f}$ is continuous. We want to show that, for any open $U \subseteq Z$ and any $(x, y) \in f^{-1}(U)$, there is a product open nbh $V \times W$ of (x, y) in $f^{-1}(U)$.

Since $\widehat{f}(x) \in \mathrm{Map}(Y, Z)$, there is an open nbh V of y so that $\overline{V}$ is compact and contained in $\widehat{f}^{-1}(U)$. But then $\widehat{f}(x) \in \left\langle \overline{V}, U \right\rangle$. So, there is an open nbh W of x in X so that $\widehat{f}(W) \subseteq \left\langle \overline{V}, U \right\rangle$. But this implies that $f(W \times V) \subseteq U$ as required. $\square$

7.1.1. *pointed spaces.* We need a fancier version of this. First we go to pointed spaces.

Definition 7.6. A *pointed space* is a space X with a base point $*$. A *pointed map* is a map of pointed spaces which preserves the base point. A *pointed homotopy* $h : X \times I \to Y$ keeps the base point at the base point: $h(x, *) = *$ for all $x \in X$.

Definition 7.7. If $A \subseteq X$ then X/A is the pointed space given by taking X and smashing A to a point $*$. This means you impose the relation $x \sim y$ if $x, y \in A$ or if $x = y$. You give X/A the quotient topology. So, a subspace of X/A is open iff its inverse image in X is open.

Proposition 7.8. *If X is regular and A is closed then X/A is Hausdorff.*

Proof. If $x \neq A$ then there are disjoint open nbhs U, V of x, A in X. Then $q(U), q(V)$ are open nbhs of $q(x), *$ in X/A which separate these points. Here $q : X \to X/A$ is the quotient map. There is no problem separating other points. □

Definition 7.9. The *smash product* $X \wedge Y$ of two pointed spaces is defined by:

$$X \wedge Y = \frac{X \times Y}{X \times * \cup * \times Y}$$

Corollary 7.10. *If X, Y, Z are pointed spaces and Y is locally compact then a pointed map $f : X \wedge Y \to Z$ is continuous if and only if its adjoint $\widehat{f} : X \to \mathrm{Map}_0(Y, Z)$ is continuous.*

This just follows from the theorem and the definitions.

Definition 7.11. $\mathrm{Map}_0(Y, Z)$ is the space of all pointed maps $Y \to Z$ with the compact-open topology. This is a subspace of $\mathrm{Map}(Y, Z)$. The base point of $\mathrm{Map}_0(Y, Z)$ is the mapping $*$ which sends all of Y to the base point of Z.

7.1.2. *fancy version of adjunction formula.*

Corollary 7.12. *Suppose that X, Y are locally compact Hausdorff spaces. Then adjunction gives a homeomorphism:*

$$\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y, Z))$$

The theorem says that, if Y is locally compact, we have a bijection of sets. We need to show that this bijection is a homeomorphism. We use the following easy principle.

Lemma 7.13. *Suppose that $f : X \to X'$ is a bijection of the underlying sets of topological spaces X, X'. Suppose that for all space W the mapping $\varphi : W \to X$ is continuous if and only if $\psi = f \circ \varphi : W \to X'$ is continuous. Then f is a homeomorphism.*

Proof. Let $W = X$ and $\varphi = id_X$. This is continuous. So, $\psi = f$ is continuous. Let $W = X'$ and $\varphi = f^{-1}$. Then $\psi = id_{X'}$ is continuous. So, f^{-1} is continuous. So, f is a homeomorphism. □

Proof of Corollary. Let W be any space and let

$$\varphi : W \to \mathrm{Map}(X \times Y, Z)$$

be any set mapping. Then, since $X \times Y$ is locally compact, the theorem tells us that φ is continuous if and only if its adjoint

$$\widehat{\varphi} : X \times Y \times W \to Z$$

is continuous. But Y is locally compact, so this is continuous iff its adjoint

$$X \times W \to \mathrm{Map}(Y, Z)$$

is continuous. But X is locally compact. So, this is continous iff

$$\psi : W \to \mathrm{Map}(X, \mathrm{Map}(Y, Z))$$

is continuous. By the lemma this implies that the adjunction bijection

$$\mathrm{Map}(X \times Y, Z) \to \mathrm{Map}(X, \mathrm{Map}(Y, Z))$$

is a homeomorphism. □

Corollary 7.14. *If X, Y, Z are pointed and X, Y are locally compact then adjunction gives a homeomorphism*

$$\mathrm{Map}_0(X \wedge Y, Z) \cong \mathrm{Map}_0(X, \mathrm{Map}_0(Y, Z))$$

7.1.3. *homotopy groups.* I used adjunction to define the product structure on homotopy groups and I gave two proofs that higher homotopy groups are abelian.

7.2. **Fibrations.**

Definition 7.15. A *fibration* is a continuous mapping $p : E \to B$ with the *homotopy lifting property.* This means that, for any mapping $f : X \to E$ and any homotopy $h : X \times I \to B$ of $p \circ f : X \to B$, there exists a lifting $\widetilde{h}$ of h to E, i.e., a mapping $\widetilde{h} : X \times I \to E$ of h to E making the following diagram commute. (Here $i_0 : X \to X \times I$ is the mapping $i_0(x) = (x, 0)$.)

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "h"] \arrow[ur, dashed, "\exists \widetilde{h}"] & B \end{tikzcd}$$

7.2.1. *path-loop fibration.* The standard example of a fibration is the *path-loop fibration* given as follows. For any pointed space X let PX be the space of all paths in X starting at the base point:

$$PX = \{\gamma : I \to X \mid \gamma(0) = *\} = \mathrm{Map}_0((I, 0), (X, *))$$

Let $p : PX \to X$ be evaluation at 1. I.e., $p(\gamma) = \gamma(1)$. The inverse image of $*$ is the *loop space* of X:

$$\Omega X = \{\gamma : I \to X \mid \gamma(0) = \gamma(1) = *\}$$

Theorem 7.16. *$p : PX \to X$ is a fibration.*

Proof. We have the following diagram:

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "i_0"'] & PX \arrow[d, "p"] \\ Z \times I \arrow[r, "h"] \arrow[ur, dashed, "\exists \widetilde{h}"] & X \end{tikzcd}$$

where we need to find the lifting $\widetilde{h} : Z \times I \to PX$.

First, let's look at what is given: $p \circ f = h \circ i_0$. In terms of the adjoint $\widehat{f} : Z \times I \to X$ and h this condition is that $\widehat{f}(z, 1) = h(z, 0)$. So, we can paste these together to form a continuous mapping $H : Z \times [0, 2] \to X$ by

$$H(z, t) = \begin{cases} \widehat{f}(z, t) & \text{if } t \leq 1 \\ h(z, t-1) & \text{if } t \geq 1 \end{cases}$$

Note that, since $f(z) \in PX$, $\widehat{f}(z, 0) = *$ for all $z \in Z$.

The lifting $\widetilde{h}$ should be a continuous mapping satisfying the following two conditions.

(1) $p \circ \widetilde{h} = h$, i.e., $\widetilde{h}(z, t)(1) = h(z, t)$ for all $(z, t) \in Z \times I$.
(2) $\widetilde{h} \circ i_0 = f$, i.e., $\widetilde{h}(z, 0) = f(z)$

Since I is locally compact, the existence of such a map is equivalent to the existence of its adjoint $\widehat{\widetilde{h}} : Z \times I \times I \to X$ satisfying the following three conditions.

(1) $\widehat{\widetilde{h}}(z, s, 1) = h(z, s)$
(2) $\widehat{\widetilde{h}}(z, 0, t) = \widehat{f}(z, t)$ where $\widehat{f} : Z \times I \to X$ is the adjoint of $f : Z \to P(X)$
(3) $\widehat{\widetilde{h}}(z, s, 0) = *$

In terms the square I^2, the condition is that $\widehat{\widehat{h}}$ is equal to h on the top edge, $\widehat{f}$ on the left edge and $*$ on the bottom edge. A function with these properties is given by

$$\widehat{\widehat{h}}(z,s,t) = H(z, t(s+1))$$

$\square$

7.2.2. *pull-back.* We need to know that any pull-back of a fibration is a fibration. But first we need the definition of the pull-back:

Definition 7.17. Suppose that $f : X \to B$ and $p : E \to B$ are any two continuous maps. Then the *pull-back* f^*E of E along f is defined to be the universal object making the following diagram commute:

$$\begin{tikzcd}
f^*E \arrow[r, "h"] \arrow[d, "g"'] & E \arrow[d, "p"] \\
X \arrow[r, "f"] & B
\end{tikzcd}$$

By universal I mean that, for any space Z and maps $\alpha : Z \to X, \beta : Z \to E$ so that $f \circ \alpha = p \circ \beta$, there exists a unique mapping $\gamma : Z \to f^*E$ making the following diagram commute:

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "\alpha"'] & & \\
& f^*E \arrow[r, "h"] \arrow[d, "g"'] & E \arrow[d, "p"] \\
& X \arrow[r, "f"] & B
\end{tikzcd}$$

The explicit formula for the pull-back is

$$f^*E = \{(x,e) \in X \times E \mid f(x) = p(e)\}$$

Then the unique mapping $Z \to f^*E$ making the diagram commute is (α, β).

Lemma 7.18. *Every pull-back of a fibration is a fibration.*

Proof. This is a diagram argument which works in any category:

$$\begin{tikzcd}
Z \arrow[r] \arrow[d, "i_0"'] & f^*E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
Z \times I \arrow[r, "h"] \arrow[ur, dashed, "\exists"] & X \arrow[r, "f"] & B
\end{tikzcd}$$

Since $p : E \to B$ is a fibration and the outside rectangle commutes, there exists a mapping $\widetilde{h} : Z \times I \to E$ making the diagram commute. Then $(h, \widetilde{h}) : Z \times I \to f^*E$ is the mapping that we want. $\square$

If we apply this to the path-loop fibration $PX \to X$ and any mapping $f : A \to X$ we get a fibration $f^*PX \to A$ where

$$f^*PX = \{(a, \gamma) \in A \times PX \mid \gamma(1) = f(a)\}$$

In the special case $A \subseteq X$ we get the *relative path space*

$$P(X, A) = \{\gamma : I \to X \mid \gamma(0) = *, \gamma(1) \in A\}$$

which is a fibration over A with mapping $P(X, A) \to A$ given by $\gamma \mapsto \gamma(1)$. The fiber is the loop space of X:

$$\begin{tikzcd} \Omega X \arrow[r] & P(X,A) \arrow[d] \\ & A \end{tikzcd}$$

7.2.3. *converting any map to a fibration.*

Theorem 7.19. *Any continuous map $f : X \to Y$ is equivalent to a fibration.*

Proof. Let $E = \{(x, \gamma) \in X \times Y^I \,|\, \gamma(1) = f(x)\}$. Then I claim that $p_0 : E \to Y$ given by $p_0(x, \gamma) = \gamma(0)$ is a fibration. In other words, there is a lifting $\widetilde{h} : Z \times I \to E$ making the following diagram commute:

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "i_0"'] & E \arrow[d, "p_0"] \\ Z \times I \arrow[r, "h"] \arrow[ur, dashed, "\exists \widetilde{h}"] & Y \end{tikzcd}$$

Since $E \subseteq X \times Y^I$, $g(z)$ has two coordinates: $g(z) = (x_z, \gamma_z)$. The definition of the pull-back E is that $f(x_z) = \gamma_z(1)$. The mapping $h : Z \times I \to Y$ satisfies the condition $h(z, 0) = p_0 g(z) = \gamma_z(0)$. The lifting $\widetilde{h} : Z \times I \times I \to E$ has first coordinate given by $(z, s, t) \mapsto x_z$ and second coordinate given somehow so that, on $I \times I$, the mapping is γ_z on the left side, $f(x_z)$ on the top and h on the bottom. Since this "C" shaped subset is a retract of the square, there is a way to fill in the rest of the square.

Since the only restriction on the second coordinate γ of $(x, \gamma) \in E$ is that $\gamma(1) = f(x)$, we can contract γ to the point $f(x)$ and we have a deformation retraction of E to the space of pairs $(x, f(x))$ where $f(x)$ is the constant path at $f(x)$. This means we have a homotopy equivalence $X \simeq E$ given by $x \mapsto (x, f(x))$. When we say that $f : X \to Y$ is equivalent to a fibration we mean that we have a commuting diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\simeq"'] & Y \arrow[d, "="] \\ E \arrow[r, "p_0"] & Y \end{tikzcd}$$

where $p_0 : E \to Y$ is a fibration and the vertical maps are homotopy equivalences. $\square$

7.3. **Cofibrations.** Cofibrations are the dual concept to fibrations. The main theorem that we need is that the inclusion of any subcomplex $A \hookrightarrow X$ of a CW complex X is a cofibration.

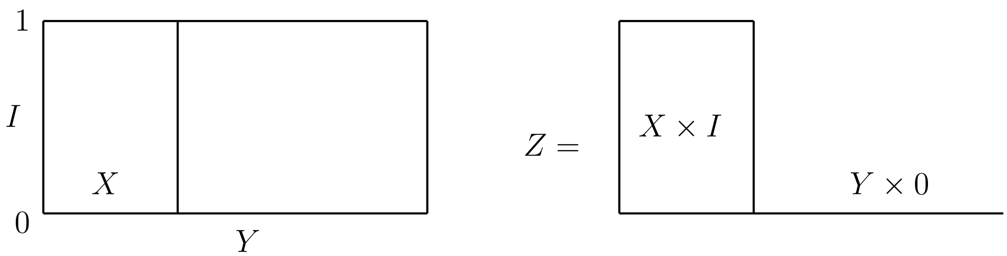
Definition 7.20. $j : X \to Y$ is a *cofibration* if it has the *homotopy extension property* (HEP). This says that, for any continuous mapping $f : Y \to Z$, any homotopy h of $f \circ j : X \to Z$ extends to a homotopy of f as shown in the following diagram.

$$\begin{tikzcd}
X \arrow[r, "i_0"] \arrow[d, "j"'] & X \times I \arrow[d, "j \times id_I"'] \arrow[ddr, bend left, "h"] & \\
Y \arrow[r, "i_0"'] \arrow[drr, bend right, "f"'] & Y \times I \arrow[dr, dashed, "\exists \overline{h}"] & \\
 & & Z
\end{tikzcd}$$

I use the notation $X \rightarrowtail Y$ to indicate a cofibration.

It follows from the definition that a cofibration is an embedding, i.e. a homeomorphism onto its image. If Y is Hausdorff, then X must be a closed subset of Y. Instead of proving these, I will just assume that these additional conditions hold.

There is a *universal* choice for Z, namely, $Z = X \times I \cup Y \times 0$ with $f = i_0 : Y \to Z$ and $h : X \times I \to Z$ being the inclusion map. If the homotopy h extends to a homotopy $\overline{h}$ of the inclusion $f = i_0 : Y \to Z$ then $\overline{h}$ is a retraction of $Y \times I$ to Z.



Theorem 7.21. *The following are equivalent.*

(1) $X \hookrightarrow Y$ *is a cofibration.*
(2) $X \times I \cup Y \times 0$ *is a retract of* $Y \times I$.
(3) $X \times I \cup Y \times 0$ *is a deformation retract of* $Y \times I$.

Proof. It is obvious that (1) and (2) are equivalent and that (3) implies (2). The only question is why do (1) and (2) imply (3)?

Suppose that $r : Y \times I \to Z = X \times I \cup Y \times 0$ is a retraction of $Y \times I$ to Z and $r = (r_1, r_2)$. Then we can transform r into a deformation retraction just by doing it slowly:

$$H(x, t, u) = (r_1(x, tu), t(1 - u) + ur_2(x, t))$$

□

There is an intuitive description of what cofibrations are, or at least what is a sufficient conditions for having a cofibration: You can pull a neighborhood of X into X. This means you have a "distance" function $\phi : Y \to I$ which is 0 only on X and equal to 2 outside a neighborhood and a homotopy $h : Y \times I \to Y$ of the identity map of Y which pulls points into X at time at most the distance: $h(y, \phi(y)) \in X$. We also assume that, when you reach X you stay there: $h(y, t) = h(y, \phi(y))$ if $t \geq \phi(y)$. In particular $h(x, t) = x$ if $x \in X$.

Such a homotopy h produces a retraction r of $Y \times I$ into $Z = X \times I \cup Y \times 0$ by

$$r(y,t) = (h(y,t), max(0, t - \phi(y)))$$

7.3.1. *mapping cylinders and cones.* Standard examples of cofibrations are mapping cylinders and mapping cones.

Definition 7.22. The *mapping cylinder* $Z(f)$ of a continuous mapping $f : X \to Y$ is given by attaching the cylinder $X \times I$ to Y along the map f giving a space which looks like a magician's hat:

$$Z(f) = X \times I \coprod Y/(x,0) \sim f(x)$$

It is easy to see that the inclusions $Y = Y \times 0 \rightarrowtail Z(f)$ and $X = X \times 1 \rightarrowtail Z(f)$ are cofibrations since we can pull neighborhoods into these subsets. There is also a categorical proof given below.

Definition 7.23. The *mapping cone* $C(f)$ of a continuous mapping $f : X \to Y$ is given by pinching the top of the mapping cylinder to a point:

$$C(f) = Z(f)/X \times 1$$

The inclusion of the base of the mapping cone is still a cofibration: $Y \rightarrowtail C(f)$.

One very important example is given by attaching a cell to a space:

$$A \rightarrowtail A \cup e^n$$

This is the mapping cone of the attaching map $\eta : S^{n-1} \to A$. Thus $A \cup_\eta e^n = C(\eta)$.

Before going on to cell complexes, I want to do the push-out construction.

Definition 7.24. Given two mappings $f : A \to Y$ and $j : A \to X$, the *push-out* $X \cup_f Y$ is universal object making the following diagram commute:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "j"'] & Y \arrow[d] \\
X \arrow[r] & X \cup_f Y
\end{tikzcd}$$

I.e., given any other space Z and maps $\alpha : X \to Z$ and $\beta : Y \to Z$ so that $\alpha \circ j = \beta \circ f$ there is a unique map $X \cup_f Y \to Z$ making the following diagram commute:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "j"'] & Y \arrow[d] \arrow[ddr, bend left, "\beta"] & \\
X \arrow[r] \arrow[drr, bend right, "\alpha"'] & X \cup_f Y \arrow[dr, dashed, "\exists!"] & \\
 & & Z
\end{tikzcd}$$

The formula for the push-out is simply the quotient space $X \cup_f Y = X \coprod Y/ \sim$ where the equivalence relation is $f(a) \sim j(a)$ for all $a \in A$. The unique mapping $X \cup_f Y \to Z$ is just $\alpha \cup \beta$.

Lemma 7.25. *Any push-out of a cofibration is a cofibration.*

Proof. To prove this we needed to rephrase the definition of a cofibration so that it looks like the definition of a fibration with arrows reversed. Then the proof of this lemma becomes the same as the proof of the dual statement that any pull-back of a fibration is a fibration.

So, here is the new diagramatic version of the definition of a cofibration:

$$\begin{tikzcd} X \arrow[r, "\widehat{h}"] \arrow[d, "j"'] & Z^I \arrow[d, "ev_0"] \\ Y \arrow[r, "f"'] \arrow[ur, dashed, "\exists \widehat{\widehat{h}}"] & Z \end{tikzcd}$$

ev_0 is evaluation at 0 ($ev_0(\gamma) = \gamma(0)$) and $\widehat{h} : X \to Z^I$ is the adjoint of a homotopy of $f \circ j : X \to Z$. Now, if we push-out the cofibration $j : A \rightarrowtail X$ along a map $f : A \to Y$ and we want to show that the resulting map $Y \to X \cup_f Y$ is a cofibration we get the following diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & Y \arrow[r] \arrow[d] & Z^I \arrow[d, "ev_0"] \\ X \arrow[r] & X \cup_f Y \arrow[r, "h"'] \arrow[ur, dashed, "\exists"] & Z \end{tikzcd}$$

Since $A \to X$ is a cofibration the mapping $X \to Z$ lifts to Z^I. But then $X \cup_f Y$ is the push-out of the left hand square so there is a unique induced map $X \cup_f Y \to Z^I$ as required. $\square$

Lemma 7.26. *The inclusion map $X \times 0 \rightarrowtail X \times I$ is a cofibration.*

Proof. We need to show that $X \times 0 \times I \cup X \times I \times 0$ is a retract of $X \times I \times I$. But that follows from the fact that $0 \times I \cup I \times 0$ is a retract of $I \times I$. $\square$

Lemma 7.27. *The inclusion of both the top and bottom of the mapping cylinder $X \to Z(f)$ and $Y \to Z(f)$ are cofibrations.*

Proof. Since the mapping cylinder $Z(f)$ of a mapping $f : X \to Y$ is the push-out in the diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_0"'] & Y \arrow[d] \\ X \times I \arrow[r] & Z(f) \end{tikzcd}$$

and $i_0 : X \to X \times I$ is a cofibration, so is $Y \to Z(f)$.

To show that $X \to Z(f)$ is a cofibration we need to find a retraction of $Z(f) \times I$ to $X \times I \cup Z(f) \times 0$. Such a retraction r is given by composing a retraction of $X \times I$ onto $X \times (I \times 0 \cup 0 \times I \cup 1 \times I)$ (which is the identity on $Y \times I$) with the projection map $Y \times I \to Y$. $\square$

Theorem 7.28. *Any continuous mapping $f : X \to Y$ is homotopy equivalent to a cofibration.*

Proof. We can express f as a composition of the cofibration $X \rightarrowtail Z(f)$ and the retraction $p : Z(f) \to Y$ given by the composition of the projection map $X \times I \to X$ with the map $f : X \to Y$ (and the identity on Y. Since $p : Z(f) \xrightarrow{\simeq} Y$ is a homotopy equivalence, the map $f : X \to Y$ is homotopy equivalent to the cofibration $X \to Z(f)$. $\square$

7.4. **Cell complexes.** As I pointed out earlier, one example of a cofibration is the inclusion of the base of a mapping cone $Y \rightarrowtail C(f : X \to Y)$ and one example of a mapping cone is given by attaching a cell.

7.4.1. *attaching one cell.*

Definition 7.29. An *n-cell* $e = e^n$ in a Hausdorff space Z is a subspace of Z satisfying the following.

(1) $e = e^n$ is relatively open, i.e., $e = X \cap U$ where X is a closed subset of Z and U is an open subset of Z. Equivalently, $U = Z - A$ where A is a closed subset of Z disjoint from e^n and $X = A \cup e^n$.

(2) e^n is homeomorphic to an open n-ball. Thus we have a homeomorphism:

$$\psi : B_1(0) \xrightarrow{\cong} e^n$$

(3) This homeomorphism extends to a continuous mapping on the unit disk, the closed n-ball in $\mathbb{R}^n$:

$$\overline{\psi} : \overline{B}_1(0) = D^n \to X.$$

(4) The restriction of $\overline{\psi}$ to the unit sphere:

$$\partial D^n = S^{n-1} \xrightarrow{\eta} A$$

$\eta = \overline{\psi}|S^{n-1}$ is called the *attaching map* of the n-cell e^n.

Exactly how much of this information is specified is a little ambiguous. If we change A or ψ or $\overline{\psi}$ do we have the "same" cell? I'll take the point of view that the "cell" is the subspace $e \subseteq Z$ but it comes with all this additional stuff which we are free to change if we wish.

Proposition 7.30. *$X = A \cup e^n$ is homeomorphic to the mapping cone $C(\eta)$ of the attaching map $\eta : S^{n-1} \to A$ of the cell e^n.*

Proof. We have two continuous mapping: the inclusion $A \hookrightarrow X$ and the map $\overline{\psi} : D^n \to X$. These induce a mapping on the push-out $A \cup_\eta D^n = C(\eta)$:

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d, "\eta"'] & D^n \arrow[d] \arrow[ddr, bend left, "\overline{\psi}"] & \\
A \arrow[r] \arrow[drr, bend right, "inc"'] & A \cup_\eta D^n \arrow[dr, dashed, "h"] & \\
 & & X
\end{tikzcd}$$

Thus we get a continuous mapping $h : A \cup_\eta D^n = C(\eta) \to X$ which is clearly a bijection. We need to show that the inverse mapping h^{-1} is continuous, i.e., that h takes closed sets to closed sets.

Suppose that C is a closed subset of $A \cup_\eta D^n$. Then the inverse image of C in D^n is closed and thus compact. Call it K. The inverse image of C in A is also closed. Call it $C \cap A$. The image $h(C)$ of C in X is then the union of the closed set $C \cap A$ and the compact set $\overline{\psi}(K)$ and is therefore closed. (Remember that Z and thus X are assumed to be Hausdorff. So compact subsets are closed.) $\square$

Using the following trivial observation we can conclude that any finite relative cell complex is a cofibration.

Exercise 7.31. Show that any composition of cofibrations $A \rightarrowtail B, B \rightarrowtail C$ is a cofibration: $A \rightarrowtail C$. (Similarly, any composition of fibrations is a fibration.)

Definition 7.32. A *finite relative cell complex* (X, A) is a pair of Hausdorff spaces given by attaching a finite number of cells to A:

$$X = A \cup e_1^{n_1} \cup e_2^{n_2} \cup \cdots \cup e_k^{n_k}$$

where each cell is attached to the union of A and the previous cells.

By induction on k and the exersize, we have:

Corollary 7.33. *Any finite relative cell complex is a cofibration $A \rightarrowtail X$.*

If we have an infinite number of cells then we need the weak topology condition which is the W in the name CW-complex.

7.4.2. *CW-complexes.* A CW-complex is a topological space together with a cell decomposition of the space. For example, a sphere can be given the structure of a CW-complex by choosing one point as the 0-cell and the rest as an n-cell. If we change the 0-cell we get a different CW-complex. If we change the homeomorphism we have the same CW-complex with a different parametrization of the n-cell.

Definition 7.34. A *CW-complex* is a Hausdorff space X together with a decomposition into cells $X = \cup e_\alpha^n$ so that

(1) $X = \cup X^n$ where $X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots$

(2) The *n-skeleton* X^n is obtained from X^{n-1} by attaching n-cells (possibly infinitely many)
$$X^n = X^{n-1} \cup e_\alpha^n \cup e_\beta^n \cup \cdots$$

(3) All cells are attached on cells of lower dimension. Thus the attaching map of e_α^n is a mapping $\eta_\alpha : S^{n-1} \to X^{n-1}$.

(C) X is *closure finite.* This means the closure of each cell is contained in finitely cells. (Note that the word "cell" refers to the open cells not their closures.)

(W) X had the *weak topology* with respect to the closures of the cells, i.e., a subset C of X is closed if and only if $C \cap \overline{e}_\alpha^n$ is closed and thus compact for every cell e_α^n. This in turn is equivalent to saying that X has the quotient topology with respect to the mapping
$$\bigcup \overline{\psi}_\alpha : \coprod_{n,\alpha} D^n \to X$$

Exercise 7.35. Suppose that C is a subset of X so that C meets each cell e_α^n in a finite set. Then show that C has the discrete topology.

We did this in class in steps. I will just write the steps here.

(1) First show that $C \cap \overline{e_\alpha^n}$ is finite for all cells e_α^n.
(2) Conclude that C is a closed subset of X.
(3) Show that every subset of C is also closed and thus C has the discrete topology.

A *subcomplex* of a CW-complex X is a closed subset A which is a union of cells. For example, the n-skeleton X^n is a subcomplex of X. A subcomplex of X is called *finite* if it has only finitely many cells. By definition, every cell is contained in a finite subcomplex.

Theorem 7.36. *Every compact subset of a CW complex is contained in a finite subcomplex.*

Proof. Suppose not. Then the set meets infinitely many cells. Take one point from each of these cells. This gives an infinite discrete set which is a closed subset of a compact set and therefore compact. That is impossible. $\square$

Theorem 7.37. *If A is a subcomplex of a CW-complex X then the inclusion $A \rightarrowtail X$ is a cofibration.*

Proof. The proof is first by induction on dimension and then by a limit argument using the weak topology assumption.

Let $X_n = A \cup X^n$. This is the union of A with all cells of X of dimension $\leq n$. Then $X_{-1} = A$ and X_n is obtained from X_{n-1} by attaching cells of dimension n and all of these cells are attached to X_{n-1}:

$$X_n = X_{n-1} \cup e^n_\alpha \cup e^n_\beta \cup \cdots$$

I claim that $X_{n-1} \rightarrowtail X_n$ is a cofibration. To see this look at the defining diagram:

$$\begin{tikzcd}
 & X_{n-1} \arrow[r, "h"] \arrow[d] & Z^I \arrow[d] \\
X_{n-1} \cup e^n_\alpha \cup e^n_\beta \cup \cdots & = X_n \arrow[r] \arrow[ur, dashed, "\exists \overline{h}?"] & Z
\end{tikzcd}$$

Since $X_{n-1} \rightarrowtail X_{n-1} \cup e^n_\alpha$ is a cofibration for each cell e^n_α we know that the homotopy $h : X_{n-1} \to Z^I$ extends to a homotopy $h_\alpha : X_{n-1} \cup e^n_\alpha \to Z^I$. The homotopy extension that we want is the union of all of these

$$\overline{h} = \bigcup_\alpha h_\alpha : X_n = \bigcup_\alpha (X_{n-1} \cup e^n_\alpha) \to Z^I$$

We can construct these extensions one step at a time giving a sequence of compatible extensions $\overline{h}_n : X_n \to Z^I$ of the original homotopy $h : A \to Z^I$. The final step in the proof is to show that the union of all of these mappings gives a continuous map

$$\overline{h} = \cup \overline{h}_n : X = \bigcup X_n \to Z^I$$

But this follows from the weak topology assumption. Any mapping on X is continuous if and only if it is continuous on every closed cell if and only if the mapping when restricted to each skeleton X^n is continuous. But $X^n \subseteq X_n$. $\square$

Here is the lemma that we used. This lemma is equivalent to the weak topology condition.

Lemma 7.38. *A mapping $f : X \to Z$ from a CW-complex X to any other space Z is continuous if and only if the composition $f \circ \overline{\psi}_\alpha : D^n \to Z$ of f with the parametrization map for every cell e^n_α is continuous.*

7.5. **Cellular approximation.** We want to study homotopy groups of CW-complexes and, more generally, we want to study homotopy classes of mapping between any two CW-complexes. The first thing we need is the cellular approximation theorem which says that any mapping of CW-complexes is homotopic to a cellular map. The key step is to prove this for the case of a single cell. The rest is an easy hand waving argument.

"Cellular" does not mean the map takes cells to cells. It takes the n-skeleton to the n-skeleton. More precisely, a map $f : X \to Y$ of CW-complexes is called *cellular* if $f(X^n) \subseteq Y^n$ for all n.

7.5.1. *homotopy groups of pairs.* First, recall the definition of homotopy groups of pairs:

$$\pi_k(X, A) = [(D^k, S^{k-1}), (X, A)]_0$$

I used adjunction to show that this is a group for $k \geq 2$:

Proposition 7.39. *$\pi_k(X, A) \cong \pi_{k-1}P(X, A) = \pi_1\Omega^{k-2}P(X, A)$ where $P(X, A)$ is the relative path space*

$$P(X, A) = \{\gamma : I \to X \mid \gamma(0) = *, \gamma(1) \in A\}$$

Consequently, $\pi_k(X, A)$ is a group for $k \geq 2$ and an abelian group for $k \geq 3$.

Theorem 7.40. *There is a long exact sequence of homotopy groups:*

$$\cdots \to \pi_k A \to \pi_k X \to \pi_k(X, A) \to \pi_{k-1}A \to \cdots$$

I explained that the third map is induced by evaluation of a path at 1:

$$\partial = ev_1 : P(X, A) \to A, \quad \gamma \mapsto \gamma(1)$$

We showed that this mapping is a fibration with fiber ΩX. So, exactness of the homotopy sequence follows from the following.

Lemma 7.41. *If $p : E \to B$ is a fibration with fiber F then we get a long exact sequence of homotopy groups:*

$$\cdots \to \pi_k F \to \pi_k E \to \pi_k B \to \pi_{k-1}F \to \cdots$$

Proof. I explained this only at the point $\pi_k E$. Suppose that we have an element of $\pi_k E$ represented by a pointed map $f : S^k \to E$. If it goes to zero in $\pi_k B$ then $p \circ f : S^k \to B$ is null homotopic. By the homotopy lifting property this null homotopy $h : S^k \times I \to B$ lifts to some homotopy $\widetilde{h}$ of f. But since $h(x, 1) = *$, $\widetilde{h}(x, 1) \in p^{-1}(*) = F$. I.e., $\widetilde{h} : f \simeq g$ where $g : S^k \to F$. So, $[f]$ is in the image of $\pi_k F$. □

For relative homotopy groups of CW-complexes, the main fact is the following.

Theorem 7.42. *If A is a subcomplex of X and all relative cells (those in X but not in A) have dimension $> k$ then $\pi_k(X, A) = 0$.*

The proof of this is in two steps. The first step is hard and the second is easy. The first step is the following lemma.

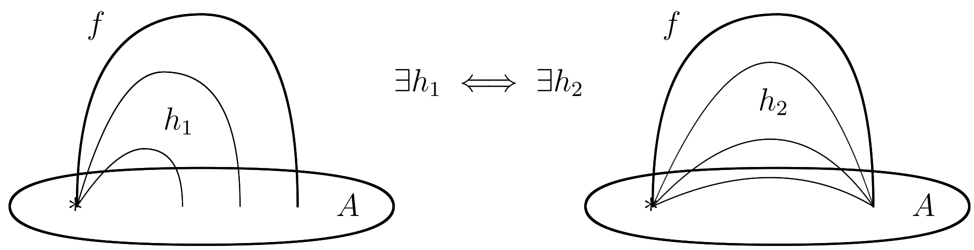
Lemma 7.43. *If $k < n$ then*

$$\pi_k(A \cup e^n, A) = 0$$

Before proving this I wanted to explain what it says and why we need it. First, what does it mean in general when we say that an element of the relative homotopy group is trivial? We have a mapping

$$f : (D^k, S^{k-1}) \to (X, A)$$

representing $[f] \in \pi_k(X, A)$. Then I explained that there are two interpretations of what it means that this element is trivial and why these are equivalent. The figure shows the case



$k = 1$. Then $f : (I, 1, 0) \to (X, A, *)$ is a path in X which starts at the base point $* \in A$, wanders around in X and ends at a point $f(1) \in A$. The first null homotopy h_1 follows the definition. If $f \simeq *$ then $f(D^k)$ will slowly deform to the basepoint and $f(S^{k-1}) \subseteq A$ will also slide to the basepoint inside A.

The second null homotopy h_2 does not actually end up at the base point. This homotopy fixes the boundary. In other words, $h_2(x, t) = f(x)$ for all $x \in \partial D^k$ and all $t \in I$. What it does is to deform the rest of $f(D^k)$ into A. It pushes the image "down" into A.

I claim that these are "equivalent." By this I mean the existence of a null homotopy of one kind implies the existence of a null homotopy of the other kind.

($\Leftarrow$) Suppose that h_2 exists. Then we can deform f into A. After that we can contract it inside A since any mapping $D^k \to A$ is null homotopic.

($\Rightarrow$) If h_1 exists then its image sweeps out a $k+1$ dimensional half-disk which can be reparametrized to give a deformation of type h_2. An explicit formula can be given as follows:

$$h_2(x, t) = \begin{cases} h_1(\frac{1}{t}x, t) & \text{if } ||x|| \le t \\ h_1(\frac{1}{||x||}x, ||x||) & \text{if } ||x|| \ge t \end{cases}$$

There is a continuity problem at $t = 0$ which we fix by letting h_1 go to zero faster so that $h_1(x, t) = *$ for $t < \epsilon$.

Proof of Theorem 7.42 assuming Lemma 7.43. The first step is to reduce to the finite case. Since the image of any map $f : (D^k, S^{k-1}) \to (X, A)$ is compact, it is contained in a finite subcomplex of X. So we may assume that X has only finitely many cells which are not in A.

Next we do induction on the number of relative cells m. If $m = 1$ then we are in the case of the lemma. So, we are done. Suppose that the theorem holds $m - 1$ and we want to prove it for m. Then $X = Y \cup e^n$ where $Y = A \cup e_1 \cup \cdots \cup e_{m-1}$. We have by the lemma that $\pi_k(X, Y) = 0$ and we have by induction that $\pi_k(Y, A) = 0$.

To show that $\pi_k(X, A) = 0$ we start with a mapping $f : (D^k, S^{k-1}) \to (X, A)$. Since $\pi_k(X, Y) = 0$ there is a homotopy of type h_2 which deforms $f(D^k)$ into Y while keeping $f|S^{k-1}$ fixed. But this means that $f(S^{k-1})$ stays inside A. So, $h_2 : f \simeq g$ where $g : (D^k, S^{k-1}) \to (Y, A)$. By induction on m we can deform g into A. So, these two deformations together deform f into A showing that $[f] = 0$. $\square$

7.5.2. *cellular approximation theorem.* More hype about why Lemma 7.43 is so important. We can use the Theorem 7.42 which follows from the lemma to prove the following very important theorem.

Theorem 7.44 (cellular approximation theorem)**.** *Any mapping of CW-complexes is homotopic to a cellular map.*

Proof. Suppose we have a continuous mapping $f : X \to Y$. Then we will construct, by induction on n, a mapping $g_n : X \to Y$ which is homotopic to f so that $g_n(X^m) \subseteq Y^m$ for all $m < n$ (i.e., g_n is cellular on X^n) and so that g_n, g_m agree on X^m. As I pointed out before, the weak topology on X implies that the union of a sequence of compatible mappings $g_n|X^n$ as I just described gives a continuous mapping $g = \bigcup g_n : X = \bigcup X^n \to Y$. To show that $f \simeq g$ we need to make sure that the homotopies $h_n : f \simeq g_n$ are all compatible.

Suppose by induction that we have a mapping $g_n : X \to Y$ which is cellular on X^n and a homotopy $h_n : f \simeq g_n$. Then for every $n+1$ cell e_α^{n+1} of X we get a mapping

$$g_n \circ \overline{\psi}_\alpha : (D^{n+1}, S^n) \to (X^{n+1}, X^n) \to (Y, Y^n)$$

Since $\pi_{n+1}(Y, Y^{n+1}) = 0$ by Theorem 7.42, we can deform $g_n \circ \overline{\psi}_\alpha(D^{n+1})$ into Y^{n+1} while fixing the map on S^n. Because this homotopy fixes S^n it gives a homotopy of $g_n|X^{n+1}$ into Y^{n+1} which fixes $g_n|X^n$. Since $X^{n+1} \rightarrowtail X$ is a cofibration, this homotopy extends to all of X and we get a homotopy $g_n \times g_{n+1}$ where g_{n+1} is cellular on X^{n+1}. Since the homotopy fixes X^n, $g_{n+1} = g_n$ on X^n. So, we get a compatible family of cellular mappings and homotopies. $\square$

7.6. **Proof of Key Lemma.** I will now go over my long proof of Lemma 7.43 which is a special case of the simplicial approximation theorem. The lemma that we are proving is:

$$\pi_k(A \cup e^n, A) = 0 \text{ if } n > k$$

7.6.1. *the idea.* Let's look at the simplest cases of this when $k = 0, 1$ to get the idea.

($k - 0$) Then $n \geq 1$. So, the simplest example is $k = 0, n = 1$. We are looking at $\pi_0(A \cup e^1, A)$. The 1-cell e^1 is an edge attached to A at two points. An element of $\pi_0(A \cup e^1, A)$ is represented by one point on the edge e^1 which we need to deform into A. But this is easy. We slide it down along the edge to one of its two endpoints.

($k = 1$) Then $n \geq 2$. The simplest case of this is $k = 1, n = 2$. We have a 2-cell e^2 attached to A and an element of $\pi_1(A \cup e^2, A)$ is represented by a path $\gamma : I \to A \cup e^2$ starting at $\gamma(0) = * \in A$ and ending at some other point $\gamma(1) \in A$.

Suppose first that there is one point $c \in e^2$ which is not in the image of the path γ. Then we can radially deform the path away from the point c to push it into A. More generally:

Lemma 7.45. *For any $c \in e^n$, $A \cup e^n - c \simeq A$ and therefore $\pi_k(A \cup e^n - c, A) = 0$ for all k.*

Thus, in order to prove the Key Lemma we just need to deform the mapping $f : (D^k, S^{k-1}) \to (A \cup e^n, A)$ so that it misses one point. We will do this by "straightening" the mapping f near the origin in e^n. In the case $k = 1$, we subdivide the interval $D^1 = I$ and consider only those intervals $J = [a, b]$ which are mapped entirely into e^n ($f(J) \subseteq e^n$) Then we create a new mapping f' (a path in the case $k = 1$) which is equal to f on the vertices a, b and is linear inside these intervals so that $f'(J)$ is the straight line segment connecting $f'(a) = f(a)$ to $f'(b) = f(b)$. Here is the formula:

$$f'((1 - u)a + ub) = \psi((1 - u)\psi^{-1}f(a) + u\psi^{-1}f(b))$$

where $\psi : B_1(0) \to e^n$ is the parametrization of the cell e^n. Next we use the fact that any finite number of line segments is nowhere dense in $\mathbb{R}^n$ for $n \geq 2$ and therefore misses a point arbitrarily close to the origin in e^n.

7.6.2. *Lebesgue number.* The first step is to triangulate the domain disk D^k. This means to cut it up into simplices, not triangles. We will want each simplex to be contained in a member of some open covering of D^k. To do this we need the Lebesgue number of the covering.

Definition 7.46. Suppose that X is any compact metric space and $\mathcal{U} = \{U_\alpha\}$ is an open covering of X. Then the *Lebesgue number* of the covering is a positive real number λ so that any subset of X with diameter $< \lambda$ is contained in one of the open sets U_α. (The *diameter* of a subset of a metric space is the supremum of the distance between any two of its points.)

Theorem 7.47. *Every open covering of a compact metric space has a Lebesgue number.*

Proof. This is the proof I said which I did not write down: Suppose that there is no Lebesgue number. Then for any positive integer n, $\lambda = 1/n$ does not work. So, there is some subset S_n of diameter $< 1/n$ which is not contained in any U_α. Choose a point $x_n \in S_n$. Then there is a subsequence x_{n_i} of the sequence x_n converging to some point $x_\infty \in X$. But then $x_\infty \in U_\alpha$ for some α. Since the subsets S_{n_i} have diameter $\to 0$ and converge to x_∞, for sufficiently large i, S_{n_i} will be contained in U_α which is a contradiction. $\square$

7.6.3. *the covering.* We are going to take an open covering $\{U_\alpha\}$ of $A \cup e^n$, then let $V_\alpha = f^{-1}U_\alpha$. We triangulate D^k so that the simplices have diameter less than the Lebesgue number of this covering. Then the image $f(\sigma)$ of each simplex will lie in some U_α.

Let $X = A \cup e^n$ and let $p : (X, A) \to (I, 1)$ be the distance function to the origin. (Thus $p(A) = 1$ and $p\psi(x) = ||x||$ for $x \in B_1(0)$.) Let

$$U_0 = p^{-1}[0, \tfrac{1}{2}) = \psi B_{1/2}(0)$$

$$U_1 = p^{-1}(\tfrac{1}{4}, \tfrac{3}{4}) = \bigcup_{||y||=\frac{1}{2}} \underbrace{\psi B_{1/4}(y)}_{U_y}$$

$$U_2 = p^{-1}(\tfrac{1}{2}, 1) = \bigcup_{||z||=\frac{3}{4}} \underbrace{\psi B_{1/4}(z)}_{U_z}$$

$$U_3 = p^{-1}(\tfrac{3}{4}, 1]$$

Note that $A \subseteq U_3$ is disjoint from the other open sets.

Let $V_0, V_y, V_z, V_3, V_1 = \cup V_y, V_2 = \cup V_z$ be the inverse images of U_0, U_y, etc. under the mapping $f : D^k \to X$ and let λ be the Lebesgue number of the covering of D^k by the open sets V_0, V_y, V_z, V_3. Note that $\partial D^k = S^{k-1}$ is contained in V_3. Triangulate D^k so that all the pieces with curved sides lie in V_3. Then we have nice straight pieces left over. Let K be the union of those good simplices which lie in one of the open sets V_0, V_y, V_z. Let L be the closure of the union of all the other simplices (the ones which lie in V_3). Equivalently, L is the closure of the complement $D^k - K$.

I forgot to explain how you triangulate. One way is to first cut up the space into little cubes. It is more obvious that this can be done. Then we triangulate each cube in a standard way. (Take the center of each edge, square and cube and cone off the triangulation of the boundary cutting up squares into $2 \cdot 4$ triangles, cubes into $2 \cdot 4 \cdot 6$ tetrahedra, etc.) Then K is the union of these triangulated cubes and L is the closure of the rest which we don't care about.

7.6.4. *straightening the mapping.* On the good set of simplices we can define a straightening. Let $f' : K \to e^n$ be the unique linear mapping (linear on simplices) which is equal to f on the vertex set. Note that since each of the open sets U_0, U_y, U_z is convex, if f maps a simplex σ of K into one of these open sets then $f'(\sigma)$ lies in the same open set and the linear homotopy from f to f' also keeps σ inside of its assigned open set.

Now, let $\phi : I \to I$ be any continuous mapping so that

$$\phi(t) = \begin{cases} 1 & \text{if } t \le \frac{1}{2} \\ 0 & \text{if } t \ge \frac{3}{4} \end{cases}$$

Let $u : D^k \times I \to I$ be given by

$$u(x, t) = (\phi \circ p \circ f(x))t = \begin{cases} t & \text{if } x \in V_0 \\ 0 & \text{if } x \in V_3 \end{cases}$$

Let $h : D^k \times I \to X = A \cup e^n$ be the homotopy given by

$$h(x, t) = \begin{cases} uf'(x) + (1 - u)f(x) & \text{if } x \in K \\ f(x) & \text{if } x \in L \end{cases}$$

where $u = u(x,t)$. Since $u = 0$ on $K \cap L \subseteq L \subseteq V_3$, these two mappings agree on $K \cap L$. So, h is a continuous mapping on all of $D^k \times I$. Also, if $t = 0$ then $u = 0$ and we have $h(x, 0) = f(x)$ for all $x \in D^k$. So h is a homotopy $f \simeq g$ from f to some mapping g.

Now we look at what h does on each simples $\sigma \subseteq D^k$. By construction σ lies in either V_0, V_y, V_z or V_3

Case 1 Suppose that $\sigma \subset V_0$. Then $u = t$ on $\sigma \times I$. So $h(x, 1) = g(x) = f'(x)$ for all $x \in \sigma$.

Case 2 Suppose that $\sigma \subseteq V_y$ where $||y|| = \frac{1}{2}$. Then $f(\sigma)$ and $f'(\sigma)$ both lie inside the convex set U_y. So $h(\sigma \times I) \subseteq U_y$ and $g(\sigma) \subseteq U_y$

Case 3 If $\sigma \subseteq V_z$ where $||z|| = \frac{3}{4}$. Then $f(\sigma), f'(\sigma) \subseteq U_z$ which is also convex. So $h(\sigma \times I) \subseteq U_z$ and $g(\sigma) \subseteq U_z$.

Case 4 If $\sigma \subseteq V_3$ then $u = 0$ on σ and $h = f$ on σ. So, $g(\sigma) \subseteq U_3$.

In all four cases, the homotopy h keeps σ inside the open set of the covering in which it starts.

Now, suppose that $pg(x) < \frac{1}{4}$. Then $x \in \sigma$ where σ must be in Case 1 since U_y, U_z, U_3 are all disjoint from the open ball of radius $\frac{1}{4}$ around 0. So $g(x) = f'(x)$ lies in the linear simplex $f'(\sigma)$. But there are only finitely many of these simplices so their union has measure zero in $\mathbb{R}^n$. So, if we choose almost any point $c \in B_{1/4}(0)$ then c will not be in the image of $g : D^k \to A \cup e^n$ which is homotopic to f. So, $[f] = [g] = 0$ in $\pi_k(A \cup e^n, A)$ proving the Key Lemma.

7.7. **Compactly generated topology.** The compactly generated topology fixes two problems that we encounter:

(1) The product of two CW-complexes is not always a CW-complex.
(2) Adjunction: $\text{Map}(X \times Y, Z) \cong \text{Map}(X, \text{Map}(Y, Z))$ works only for locally compact Y.

7.7.1. *definition and properties.*

Definition 7.48. Given any topological space X, we define a new topological space X_c with the same underlying set as X but with the *compactly generated topology.* This means a subset C of X_c is closed if and only if $C \cap K$ is (relatively) closed for all compact subsets K of X. If $X_c = X$ we say that X is a *k-space.*

If we add the assumption that X is Hausdorff, we can drop the word "relatively" since all compact subsets of Hausdorff spaces are closed.

Lemma 7.49. *Given any X, the identity mapping $X_c \to X$ is continuous.*

Proposition 7.50. *Every CW-complex is a k-space.*

Proof. If X is a CW-complex we need to show that the identity map $X \to X_c$ is continuous. If C is a closed subset of X_c then $C \cap \overline{e}$ is closed for cells e. This implies that C is closed in X since X has the weak topology. □

One of the main problems being addressed by the compactly generated topology is the product. So, we define a new product:

$$X \times_c Y := (X \times Y)_c$$

Theorem 7.51. *If X, Y are k-spaces then $X \times_c Y$ is the product of X and Y in the category of k-spaces and all continuous maps.*

As John pointed out, there is an elementary functorial proof of this which we will go over later.

7.7.2. *products of CW-complexes.* If X, Y are CW-complexes then what about the product $X \times Y$?

For every cell e_α^n in X and e_β^m in Y the product $e^n \times e^m$ is a cell in $X \times Y$ being homeomorphic to a product of open balls. So, $X \times Y$ is decomposed as a disjoint union of these product cells. We also have parametrizations $\overline{\psi}_\alpha : D^n \to \overline{e}_\alpha^n, \overline{\psi}_\beta : D^m \to \overline{e}_\beta^m$ giving a parametrization of the product cell

$$\overline{\psi}_a \times \overline{\psi}_\beta : D^n \times D^m \to \overline{e}_\alpha^n \times \overline{e}_\beta^m \subseteq X \times Y$$

If we choose a homeomorphism $D^{n+m} \cong D^n \times D^m$ then we get standard round disk parametrizations $\overline{\psi}_{\alpha\beta} : D^{n+m} \to X \times Y$ for all of the cells of $X \times Y$ and the closure finite condition is clearly satisfied. The only problem is that $X \times Y$ may not have the weak topology. Therefore, we can make it into a CW-complex by taking the compactly generated topology. I.e., $X \times_c Y$ is a CW-complex.

Theorem 7.52. *If X, Y are CW-complexes and either X or Y is locally compact then $X \times Y = X \times_c Y$ is a CW-complex.*

Remark 7.53. A CW-complex is locally compact if and only if every point has a neighborhood which is contained in a finite subcomplex.

Proof. We need to show that the identity map $X \times Y \to X \times_c Y$ is continuous since we already know that the map the other way is continuous. Since Y is locally compact Hausdorff, the continuity of this map is equivalent to the continuity of the adjoint map

$$X \to \mathrm{Map}(Y, X \times_c Y)$$

Since X has the weak topology this mapping is continuous if and only if the composition

$$\overline{\psi}_\alpha : D^n \to X \to \mathrm{Map}(Y, X \times_c Y)$$

is continuous for every cell e^n_α in X. Using again the assumption that Y is locally compact, the continuity of these maps is equivalent to the continuity of their adjoints:

$$\overline{\psi}_\alpha \times 1_Y : D^n \times Y \to X \times_c Y$$

Since D^n is compact, this is equivalent to the continuity of the adjoint map

$$Y \to \mathrm{Map}(D^n, X \times_c Y)$$

But this in turn is equivalent to the continuity of the composition

$$\overline{\psi}_\beta : D^m \to Y \to \mathrm{Map}(D^n, X \times_c Y)$$

Finally, this is equivalent to the continuity of the maps

$$\overline{\psi}_\alpha \times \overline{\psi}_b : D^n \times D^m \to X \times_c Y$$

But these maps are continuous since $D^n \times D^m$ is compact. $\square$

Using the following lemma we generalized this theorem to the case when X, Y are Hausdorff k-spaces.

Lemma 7.54. *If X is any k-space then a mapping $X \to Z$ is continuous if and only if its restriction to every compact subset of X is continuous.*

Theorem 7.55. *Suppose that X, Y are k-space, both are Hausdorff and at least one is locally compact. Then $X \times_c Y = X \times Y$.*