2.5. tensor product.

2.5.1. definition and basic properties. The best description of the tensor product is given by the universal property. But, we need to have a concrete description so that we can distinguish between equal and isomorphic modules.

Given modules $M, N, P$ a mapping $f : M \times N \to P$ is called\textbf{bilinear} if for all $x \in M$ and $y \in N$ the mappings

$$f(x, -) : N \to P$$

$$f(-, y) : M \to P$$

are homomorphisms of $A$-modules.

\textbf{Proposition 2.13} (A-M 2.12). For any two modules $M, N$ there is a module $T$ and a bilinear mapping $f : M \times N \to T$ which has the property that, for any other bilinear mapping $g : M \times N \to P$ there is a unique homomorphism $h : T \to P$ so that $g \circ h = f$.

\[
\begin{array}{ccc}
X \times Y & \overset{f}{\longrightarrow} & T \\
\downarrow{g} & \text{exists} & \downarrow{h} \\
Y & \longrightarrow & P
\end{array}
\]

Furthermore, $T$ is unique up to isomorphism.

The uniqueness of $T$ up to isomorphism is clear. Existence is given by the following construction.

\textbf{Definition 2.14}. If $M, N$ are modules, then let $M \otimes N$ be the $A$-module constructed as follows. Take the free module on the set $M \times N$. This is $C = A^{(M \times N)}$. Let $D$ be the submodule of $C$ generated by all elements of the form

$$ax + by, z - a(x, z) - b(y, z)$$

$$axz + bw - a(x, z) - b(x, w)$$

for all $x, y \in M$, $z, w \in N$ and $a, b \in A$. Then the quotient $C/D$ is called the\textbf{ tensor product} of $M$ and $N$ and denoted by $M \otimes N$. The image of $(x, y) \in M \times N$ in $M \otimes N$ is denoted $x \otimes y$.

\textbf{Proof of Proposition 2.13}. The mapping $f : M \times N \to M \otimes N$ given by $f(x, y) = x \otimes y$ is clearly bilinear and any mapping $g : M \times N \to P$ induces a unique mapping $\tilde{g} : C \to P$ and $g$ is bilinear iff $D \subseteq \ker \tilde{g}$. So, we get an induced map $h : C/D = M \otimes N \to P$. The map is unique since the elements $x \otimes y$ generate $M \otimes N$ and $h$ must send $x \otimes y$ to $f(x, y)$.

\qed
The tensor product has the important functorial property that, given homomorphisms \( f : M \to M' \) and \( g : N \to N' \) there is an induced homomorphism

\[ f \otimes g : M \otimes N \to M' \otimes N' \]

given by \((f \otimes g)(x \otimes y) = f(x) \otimes g(y)\). (Since \( f(x) \otimes g(y) \) is bilinear in \((x, y)\) we get this induced map.)

**Proposition 2.15.** We have natural isomorphisms

1. \( M \otimes N \cong N \otimes M \).
2. \( M \otimes (N \otimes P) \cong (M \otimes N) \otimes P \).
3. \( (M \oplus N) \otimes P \cong (M \otimes P) \oplus (N \otimes P) \).
4. \( A \otimes M \cong M \).

**Natural isomorphism** means that the following diagram commutes (in the first case)

\[
\begin{array}{ccc}
M \otimes N & \xrightarrow{\cong} & N \otimes M \\
\downarrow f \otimes g & & \downarrow g \otimes f \\
M' \otimes N' & \xrightarrow{\cong} & N' \otimes M'
\end{array}
\]

Problem: If \( a, b \) are coprime show that \( A/a \otimes A/b = 0 \).

### 2.5.2. Homological Properties

The basic homological properties of tensor product are

1. tensor product is *right exact* and
2. tensor product is *left adjoint* to \( \text{Hom} \).

The second property implies that first (and also implies that \( \text{Hom} \) is left exact). However, I want to start by stating the first property.

**Proposition 2.16 (A-M 2.18).** Suppose that \( M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0 \) is an exact sequence of \( A \)-modules and \( N \) is another \( A \)-module. Then

\[ M' \otimes N \xrightarrow{f \otimes 1} M \otimes N \xrightarrow{g \otimes 1} M'' \otimes N \to 0 \]

is exact.

**Proof.** The fact that \( g \otimes 1 : M \otimes N \to M'' \otimes N \) is onto is obvious. However, exactness at the other point is not obvious. What we do is to show that \( M'' \otimes N \) is the cokernel in the diagram. This means that given any module \( P \) and homomorphism \( h : M \otimes N \to P \) so that the composition

\[
M' \otimes N \xrightarrow{f \otimes 1} M \otimes N \xrightarrow{g \otimes 1} M'' \otimes N \xrightarrow{h} P
\]

is zero, there is a unique way to factor the map \( h \) through \( M'' \otimes N \).
Lemma 2.17. There is a natural isomorphism
\[ \text{Hom}(M \otimes N, P) \cong \text{Hom}(M, \text{Hom}(N, P)) \]
given by sending \( h : M \otimes N \to P \) to \( \hat{h} \) where \( \hat{h}(x) = h(x \otimes -) : N \to P \) for all \( x \in M \).

Assuming that the lemma is true, we continue the proof of the proposition. First, we claim that \( \hat{h} \circ f = 0 : M' \to M \to \text{Hom}(N, P) \). This is a simple calculation:
\[ \hat{h}(f(x)) = h(f(x) \otimes -) = 0 \]
Therefore \( \hat{h} \) factors uniquely through \( M'' \):

\[
\begin{array}{ccc}
M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \xrightarrow{0} & 0 \\
\downarrow{0} & & \downarrow{\hat{h}} & & \downarrow{\exists k} & & \\
\text{Hom}(N, P) & & & & \\
\end{array}
\]

Using the lemma again, the induced homomorphism \( M'' \to \text{Hom}(N, P) \) is the adjoint of a unique mapping \( k : M'' \otimes N \to P \) so that \( k \circ g \otimes 1 = 0 \). So, \((g \otimes 1, M'' \otimes N)\) is the cokernel of \( f \otimes 1 \) as claimed. \( \square \)

Proof of Lemma 2.17. The map going from right to left sends \( f : M \to \text{Hom}(N, P) \) to the map \( \hat{f} : M \otimes N \to P \) given by \( \hat{f}(x \otimes y) = f(x)(y) \). Since this is bilinear, it induces a map on the tensor product. It is straightforward to check that these two “adjunction maps” are inverse to each other. \( \square \)

Exercise 2.18. (1) Show that \( M/\text{a}M \cong A/\text{a} \otimes M \).

(2) If \( M \) is finitely generated and nonzero, then show that there is a maximal ideal \( \text{m} \) so that \( M/\text{m}M \neq 0 \).

(3) If \( A^n \cong A^m \) show that \( n = m \). (Remember that \( A/\text{m} \) is a field.)

2.5.3. change of rings. Tensor product is useful for explaining what is extension of scalars. If \( \varphi : A \to B \) is a ring homomorphism then the restriction of scalars functor is given by taking any \( B \)-module \( M \) and considering the same additive group \( M \) as an \( A \)-module with action of \( a \in A \) given by multiplication by the image of \( a \) in \( B \):
\[ a(x) = \varphi(a)x \]
We call this \( A M \). In particular \( B \) becomes an \( A \)-module \( AB \).

The extension of scalars functor takes an \( A \)-module \( M \) and gives a \( B \) module
\[ M_B = B \otimes_A M \]
(When there is more than one ring, it is helpful to put the subscript $\otimes_A$ to indicate that it is tensor product as $A$-modules.) The $B$-module structure on $M_B$ is given by multiplication on the first factor:

$$b(c \otimes x) = (bc) \otimes x$$

More generally, if $N$ is a $B$-module, we can consider $N$ as an $A$-module by restriction of scalars and we can take the tensor product $N \otimes_A M$. This is a $B$-module with action of $B$ given by multiplication on the first tensor factor.

**Proposition 2.19** (A-M 2.15). If $M, N$ are $B$-modules and $P$ is an $A$ module then

$$M \otimes_B (N \otimes_A P) \cong (M \otimes_B N) \otimes_A P$$

**Proposition 2.20** (A-M 2.17). If $M$ is a f.g. $A$-module then $M_B$ is a f.g. $B$-module.

**Proposition 2.21** (A-M 2.16). If $M$ is a f.g. $B$-module and $B$ is f.g. as an $A$-module then $AM$ is a f.g. $A$-module.

When we have a fixed ring homomorphism $\varphi : A \to B$ we say that $B$ is an algebra over $A$. If $B$ is finitely generated as an $A$-module then we say that $B$ is a finite $A$-algebra. For example $A[x]/(x^2)$ is a finite algebra over $A$. $A[x, y]$ is not a finite algebra but it is finitely generated (by $x, y$). An $A$-algebra is finitely generated if it is a quotient of a polynomial algebra $A[x_1, \cdots, x_n]$ over $A$ in finitely many variables.

2.5.4. flat modules. If $\otimes N$ is exact then $N$ is called a flat module. This means that for any short exact sequence $0 \to M' \to M \to M'' \to 0$ we get a short exact sequence

$$0 \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$$

**Exercise 2.22.** (1) $A$ is flat.

(2) If $N, N'$ are flat then $N \oplus N'$ is flat

(3) If $N, N'$ are flat then $N \otimes N'$ is flat

(4) If $N$ is flat then $\otimes N$ takes long exact sequences to long exact sequences.

(5) If $N$ is a flat $A$-module and $B$ is an algebra over $A$ then $N_B$ is a flat $B$-module.