3.4. review.
\[
\frac{x}{s} = 0 \text{ in } S^{-1}M \iff xu = 0 \text{ for some } u \in S
\]
and (Corollary 3.9):
\[
S^{-1}M \cong S^{-1}A \otimes M
\]

Lemma 3.12 (A-M 3.7). \( S^{-1}(M \otimes_A N) \cong S^{-1}M \otimes_{S^{-1}A} S^{-1}N \). In particular:
\[
(M \otimes_A N)_p \cong M_p \otimes_{A_p} N_p
\]

3.5. local properties. A property \( P \) describing a class of rings or modules is called **local** if property \( P \) holds for a ring \( A \) (or module) iff it holds for every localization \( A_p \) at a prime ideal has property \( P \). For example, the property of being equal to 0 is local:

**Proposition 3.13** (A-M 3.8). If \( M \) is an \( A \)-module, the following are equivalent.

1. \( M = 0 \)
2. \( M_p = 0 \) for every prime ideal \( p \) in \( A \)
3. \( M_m = 0 \) for every maximal ideal \( m \) in \( A \).

**Proof.** (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) is clear. So, we just need to show \( \neg(1) \Rightarrow \neg(3) \).
If \( x \neq 0 \in M \) consider \( \text{Ann}(x) \subseteq m \).

Using the exactness of localization, we can easily get:

**Corollary 3.14** (A-M 3.9). Suppose that \( \varphi : M \rightarrow N \) is a homomorphism of \( A \)-modules. Then the following are equivalent.

1. \( \varphi \) is a monomorphism.
2. \( \varphi_p : M_p \rightarrow N_p \) is a monomorphism for every prime ideal \( p \) in \( A \)
3. \( \varphi_m : M_m \rightarrow N_m \) is a monomorphism for every maximal ideal \( m \) in \( A \).

And the same holds with “monomorphism” replace with “epimorphism.”

This, and Lemma 3.12 imply that flatness is a local property:

**Proposition 3.15** (A-M 3.10). If \( M \) is an \( A \)-module, the following are equivalent.

1. \( M \) is flat.
2. \( M_p \) is a flat \( A_p \)-module for every prime ideal \( p \) in \( A \)
3. \( M_m \) is a flat \( A_m \)-module for every maximal ideal \( m \) in \( A \).
3.6. **Extended and Contracted Ideals in Localizations.** Recall: for the ring homomorphism $f : A \to S^{-1}A$ the extension of an ideal $a \subseteq A$ is given by $a^e = S^{-1}a$ and the contraction of an ideal $b \subseteq B$ is, by definition, $b^e = f^{-1}(b)$. It is clear that the contraction of a prime ideal is prime.

Also, there is always a bijection between the set of contracted ideals $C = \{b^e \mid b \subseteq B, b = b^e\}$ and the set of extended ideals $E = \{a^e \mid a \subseteq A, a = a^e\}$.

**Proposition 3.16** (A-M 3.11). 
(i) Every ideal in $S^{-1}A$ is an extended ideal.
(ii) If $a$ is an ideal in $A$ then 
$$a^{ec} = \bigcup_{s \in S} (a : s)$$

So, $a^e = S^{-1}A$ iff $S \cap a \neq \emptyset$.

(iv) $p \leftrightarrow p^e$ gives a bijection between the primes in $A$ disjoint from $S$ and the primes in $S^{-1}A$.

**Proof.** (i), (ii) are easy. For (iv): If $q$ is a prime ideal in $S^{-1}A$ then $q^e$ is prime in $A$ and $q^{ec} = q$. So, $q^e$ is disjoint from $S$ by (ii).

Going the other way, suppose that $p$ is a prime ideal in $A$ disjoint from $S$. In other words, $S \subseteq T = A - p$. Then,

$$S^{-1}T = \left\{ \frac{t}{s} \mid t \in T, s \in S \right\}$$

is a multiplicative subset of $S^{-1}A$ and

$$S^{-1}A = S^{-1}T \prod S^{-1}p$$

Therefore $p^e = S^{-1}p$ is a prime ideal in $S^{-1}A$.

**Corollary 3.17** (A-M 3.13). If $p \subseteq A$ is prime then the prime ideals of $A_p$ are $q_p$ where $q$ is a prime contained in $p$.

**Exercise 3.18.**

1. Show that the ring homomorphism $f : A \to S^{-1}A$ induces an injective mapping:

$$f^* : Spec(S^{-1}A) \to Spec(A)$$

2. If $S = \{1, h, h^2, h^3, \cdots \}$ then $S^{-1}A$ is denoted $A_h$ and the image of $Spec(A_h)$ in $Spec(A)$ is the set

$$X_h = \{ p \mid h \notin p \}$$
(3) Show that the mapping in (1) is a homeomorphism onto its image. Recall that the basic open sets in $\text{Spec}(A)$ are the subsets $X_h$ above. (The closed sets are $V(E) = \{ p \mid E \subseteq p \}$.)

**Proposition 3.19** (A-M 3.11(v)).

1. $S^{-1}(a \cap b) = S^{-1}a \cap S^{-1}b$
2. $S^{-1}(ab) = (S^{-1}a)(S^{-1}b)$
3. $S^{-1}(a + b) = S^{-1}a + S^{-1}b$
4. $S^{-1}(r(a)) = r S^{-1}a$

Putting $a = 0$ in (4) we get:

**Corollary 3.20** (A-M 3.12). $S^{-1}\text{nilrad } A = \text{nilrad } S^{-1}A$

**Lemma 3.21.** $\text{Ann}(S^{-1}(A/a)) = S^{-1}a$

**Proposition 3.22** (A-M 3.14). If $M$ is a f.g. $A$-module then

$$\text{Ann}(S^{-1}M) = S^{-1}\text{Ann}(M)$$

Since $(N : P) = \text{Ann} \left( \frac{N + P}{N} \right)$ we get:

**Corollary 3.23** (A-M 3.15). If $N, P$ are submodules of $M$ and $P$ is f.g. Then

$$S^{-1}(N : P) = (S^{-1}N : S^{-1}P)$$