7. Noetherian rings

A ring is Noetherian if it satisfied the ascending chain condition for ideal. Equivalently, every ideal is finitely generated. Equivalently, any nonempty collection of ideals has a maximal element.

**Proposition 7.1.** Any factor ring $A/\mathfrak{a}$ of a Noetherian ring is Noetherian.

**Proposition 7.2.** If $A$ is Noetherian, so is $S^{-1}A$ for any multiplicative set $S$.

**Lemma 7.3.** Any finitely generated module over a Noetherian ring is Noetherian. Any finite extension of a Noetherian ring is Noetherian.

**Theorem 7.4** (Hilbert basis theorem). If $A$ is Noetherian then so is the polynomial ring $A[x]$ and the power series ring $A[[x]]$.

**Proof.** Let $B = A[x]$ or $A[[x]]$. Then $B$ has a maximal ideal $\mathfrak{m} = (x)$ which has the following two properties in both cases:

1. $\bigcap \mathfrak{m}^n = 0$
2. The quotient $B/(\mathfrak{m}^n)$ is isomorphic to $A^n$ as $A$-module and is thus Noetherian by the lemma.

Let $\mathfrak{a}$ be any ideal in $B$. Then we claim that $\mathfrak{a} \cap \mathfrak{m}^n$ is finitely generated for $n$ sufficiently large. Since the quotient

$$\frac{\mathfrak{a}}{\mathfrak{a} \cap \mathfrak{m}^n} \subseteq B/\mathfrak{m}^n$$

is also finitely generated this will imply that $\mathfrak{a}$ is finitely generated.

Now, I restrict to the case $B = A[[x]]$. The other case is in the book and follows the same pattern.

Let $\mathfrak{b}$ be the ideal generated by all lowest degree coefficients of all elements in $\mathfrak{a}$. Let $b_1, \ldots, b_n$ be generators for $\mathfrak{b}$ and let $f_i(x) \in \mathfrak{a}$ be power series with lowest degree terms $b_i x^{m_i}$. Multiplying these by a power of $x$ we may assume that the $m_i$ are all equal to a fixed $m$. Then we claim that $f_i(x)$ generate the ideal $\mathfrak{a} \cap \mathfrak{m}^m$. To show this, take any element $g(x) \in \mathfrak{a} \cap \mathfrak{m}^m$. Let $cx^m$ be the lowest degree term in $g(x)$. Then $c \in \mathfrak{b}$. So $c = \sum c_i b_i$ for some $c_1, \ldots, c_n \in A$ so

$$g(x) - \sum c_i f_i(x) \in \mathfrak{a} \cap \mathfrak{m}^{m+1}$$

Similarly, there exist $c'_i \in A$ so that

$$g(x) - \sum c_i f_i(x) - \sum c'_i x f_i(x) \in \mathfrak{a} \cap \mathfrak{m}^{m+2}$$

Proceeding in this way we get $g(x) = \sum h_i(x) f_i(x)$ as claimed. \hfill \Box
7.0.1. Primary decomposition in Noetherian rings. An ideal \( a \) in a ring \( A \) is called \textbf{irreducible} if it is not the intersection of two strictly larger ideals.

**Lemma 7.5.** Suppose \( A \) is Noetherian. Then

1. Every ideal is a finite intersection of irreducible ideals.
2. Every irreducible ideal is primary.

**Proof.** (1) Suppose not and let \( a \) be a maximal counterexample.

(2) If \( q \) is irreducible, we need to show it is primary. By passing to \( A/q \) we may assume that \( q = 0 \). Suppose that \( xy = 0 \) and \( y \neq 0 \) then we need to show that \( x^n = 0 \) for some \( n \geq 1 \). Since \( A \) is Noetherian, the sequence of ideals \((0 : x) = \text{Ann}(x) \subseteq \text{Ann}(x^2) \cdots \) stops at some \( \text{Ann}(x^n) = \text{Ann}(x^{n+1}) \). Then I claim that

\[
(x^n) \cap (y) = 0
\]

To see this take any element \( ax^n = by \). Then

\[
ax^{n+1} = byx = 0 \Rightarrow a \in \text{Ann}(x^{n+1}) = \text{Ann}(x^n) \Rightarrow ax^n = 0
\]

Since 0 is an irreducible ideal, either \( (x^n) = 0 \) or \( (y) = 0 \). But \( y \neq 0 \). So, \( x^n = 0 \) showing that 0 is a primary ideal. \[\square\]

**Theorem 7.6.** Every ideal in a Noetherian ring has a primary decomposition.

**Lemma 7.7.** In a Noetherian ring any ideal \( a \) contains some power of its radical: \( r(a)^n \subseteq a \).

**Proof.** Suppose that \( a_i \) are generators for \( r(a) \). Then \( a_i^{m_i} \subseteq a \) for some \( m_i \). Then \( r(a)^{1+\sum(m_i-1)} \) is generated by monomials \( \prod a_i^{n_i} \) where \( \sum n_i = 1 + \sum (m_i - 1) \). So, at least one of the numbers \( n_i \geq m_i \) making each monomial an element of \( a \). \[\square\]

**Theorem 7.8.** The associated primes of an ideal \( a \) in a Noetherian ring are the prime ideals of the form \( (a : x) \) for some \( x \in A \).

**Proof.** If \( (a : x) = r(a : x) \) is prime then we know from before that it is an associated prime. Conversely, suppose that \( a = \bigcap q_i \) where \( q_i \) where \( q_i \) is \( p_i \)-primary. Let \( a_i = \bigcap_{j \neq i} q_i \). Let \( y \in a_i \), \( y \notin a \). Then

\[
a \subseteq b = (a : y) = \bigcap (q_j : y) = (q_i : y) \subseteq p
\]

So, \( r(b) = p_i \). So \( p_i^m \subseteq b \) for some \( m \geq 1 \). Take \( m \) minimal. Then \( p_i^{m-1} \nsubseteq b \). So, \( yp_i^{m-1} \subseteq a \). So \( \exists x \in yp_i^{m-1} \subseteq a_i \), \( x \notin a \). But then

\[
p_i \subseteq (a : x) = (q_i : x) \subseteq p
\]

showing that \( p_i \) has the desired form. \[\square\]