11. **Dimension theory**

The dimension of a Noetherian local ring can be defined in several ways. The main theorem is that all of these descriptions give the same number. So far we have the Krull dimension \( \dim A \) which is the length of the maximal chain of prime ideals.

11.1. **Poincaré series and Hilbert functions.** Suppose that \( \mathfrak{m} \) is a maximal ideal of \( A \)

\[
Gr(A) = \bigoplus A_n = A/\mathfrak{m} \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \mathfrak{m}^2/\mathfrak{m}^3 \oplus \cdots
\]

is the associated graded ring. If \( A \) is Noetherian then each \( A_n = \mathfrak{m}^n/\mathfrak{m}^{n+1} \) is a finite dimensional vector space over the field \( k = A_0 = A/\mathfrak{m} \) and we have the Poincaré series

\[
P_{Gr(A)}(t) = \sum_{n=0}^{\infty} t^n \dim_k A_n
\]

11.1.1. **length function.** In greater generality we consider a graded ring

\[
A = \bigoplus_{n=0}^{\infty} A_n = A_0 \oplus A_1 \oplus A_2 \oplus \cdots
\]

so that \( A \) is Noetherian and \( A_0 \) is Artinian. Suppose that

\[
M = \bigoplus_{n=0}^{\infty} M_n = M_0 \oplus M_1 \oplus M_2 \oplus \cdots
\]

is a finitely generated graded module over \( A \). Then each \( M_n \) is a finitely generated \( A_0 \)-module. Since \( A_0 \) is Artinian, each \( M_n \) has finite length \( \ell(M_n) \) and we can define the Poincaré series of \( M \) to be

\[
P_M(t) = \sum_{n=0}^{\infty} \ell(M_n) t^n
\]

Recall that the length of an \( A_0 \)-module \( N \) is defined to be the largest integer \( m = \ell(N) \) so that there is a filtration

\[
N = N_0 \supseteq N_1 \supseteq \cdots \supseteq N_m = 0
\]

by \( A_0 \)-submodules \( N_n \). It can be shown that any two maximal filtrations have the same length. The subquotients \( N_n/N_{n+1} \) are called composition factors of \( N \). They are simple modules which are unique up to isomorphism and permutation of indices.

Given this property it is easy to see that the length function satisfies the property that, for any short exact sequence of f.g. \( A_0 \)-modules

\[
0 \to M' \to M \to M'' \to 0
\]
we have \( \ell(M) = \ell(M') + \ell(M'') \). Any function on f.g. \( A_0 \)-modules with this property is called an additive function.

Some trivial observations about the Poincaré series:

1. \( 0 \leq P_M(1) \leq \infty \) in general.
2. \( P_M(1) = \sum \ell(M_n) < \infty \) iff \( M_n = 0 \) for all but finitely many \( n \).
   (We are given that \( \ell(M_n) < \infty \) for all \( n \geq 0 \).)
3. \( P_M(1) = 0 \) iff \( M = 0 \).

11.1.2. rational functions.

**Theorem 11.1.** The Poincaré series of \( M \) is a rational function of the form

\[
P_M(t) = \frac{f(t)}{\prod_{i=1}^{s}(1 - t^{k_i})}
\]

where \( f(t) \) is a polynomial in \( t \) with coefficients in \( \mathbb{Z} \).

**Proof.** The proof is by induction on \( s \), the number of homogeneous generators \( x_i \) of the ideal \( A_+ \).

If \( s = 0 \) then \( A = A_0 \) and each \( M_n \) is an \( A \)-submodule of \( M \). Since \( M \) is f.g. this implies that \( M_n = 0 \) for all but finitely many \( n \). So \( P_M(t) = f(t) = \sum \ell(M_n)t^n \) is a polynomial in \( t \) with coefficients in \( \mathbb{Z} \).

Now suppose that \( s \geq 1 \) and consider the last generator \( x_s \in A_{k_s} \). Multiplication by \( x_s \) gives a graded map \( M \to M \) of degree \( k_s \). Let \( K \) be the kernel and \( L \) the cokernel of this map. Then, for each \( n \geq 0 \) we get an exact sequence:

\[
0 \to K_{n-k_s} \to M_{n-k_s} \xrightarrow{x_s} M_n \to L_n \to 0
\]

where \( K_{n-k_s} = M_{n-k_s} = 0 \) for \( n < k_s \). Then

\[
\ell(K_{n-k_s}) - \ell(M_{n-k_s}) + \ell(M_n) - \ell(L_n) = 0
\]

Multiply by \( t^n \)

\[
\ell(K_{n-k_s})t^n - \ell(M_{n-k_s})t^n + \ell(M_n)t^n - \ell(L_n)t^n = 0
\]

Sum over all \( n \)

\[
t^{k_s}P_K(t) - t^{k_s}P_M(t) + P_M(t) - P_L(t) = 0
\]

So:

\[
P_M(t) = \frac{P_L(t) - t^{k_s}P_K(t)}{1 - t^{k_s}}
\]

Since \( x_s \) annihilates \( K \) and \( L \), these are graded modules over the graded ring \( \overline{A} = A/(x_s) \) whose positive grade ideal has \( s - 1 \) generators \( \overline{x}_1, \overline{x}_2, \ldots, \overline{x}_{s-1} \), the images of \( x_i \in A \). So, \( P_L(t) \) and \( P_K(t) \) are rational functions with denominator \( \prod_{i=1}^{s-1}(1 - t^{k_i}) \). The theorem follows. \( \square \)
There is one special case when we can easily calculate $f(t)$ by induction. This is the case in which $K = 0$, i.e., $x_s z \neq 0$ for all $z \neq 0$ in $M$. In this case we say that $x_s$ is $M$-regular. Then

$$P_M(t) = \frac{P_{M/x_s M}(t)}{1 - t^{k_s}}$$

Next, suppose that $x_{s-1}$ is $M/x_s M$-regular. Then

$$P_{M/x_s M}(t) = \frac{P_{M/x_{s-1} x_s M}(t)}{1 - t^{k_{s-1}}}$$

**Definition 11.2.** We say that (any sequence of elements of $A$) $x_k, \ldots, x_s$ is an $M$-regular sequence if this continues in the analogous way up to $x_k$. When $M = A$, we call this a regular sequence for $A$.

By induction we would get:

$$P_M(t) = \frac{P_{M/x_k x_{k+1} \cdots x_s M}(t)}{\prod_{i=k}^{s} (1 - t^{k_i})}$$

**Example 11.3.** Take $A = M = K[x_1, \ldots, x_d]$ graded in the usual way so that $x_1, \ldots, x_d$ span $A_1 = M_1$. Then $x_1, \ldots, x_d$ is an $M$-regular sequence and $M/x_1 \cdots x_d M = K$ is 1-dimensional and

$$P_M(t) = \frac{1}{(1 - t)^d} = \sum \left( \frac{n + d - 1}{d - 1} \right) t^n$$

**Proof.** (of the second equality) Start with

$$(1 - t)^{-1} = \sum t^n = 1 + t + t^2 + t^3 + \cdots$$

and differentiate $d - 1$ times to get

$$(d - 1)! (1 - t)^{-d} = \sum n(n - 1) \cdots (n - d + 2) t^{n-d+1}$$

$$(1 - t)^{-d} = \sum \left( \frac{n}{d - 1} \right) t^{n-d+1} = \sum \left( \frac{n + d - 1}{d - 1} \right) t^n$$

$$= \sum_{n=0}^{\infty} \left[ \frac{n^{d-1}}{(d-1)!} - \frac{n^{d-2}}{(d-3)!} + \cdots \right] t^n$$

11.1.3. *dimension using $P_M(t)$.*

**Definition 11.4.** Let $d = d(M)$ be the smallest positive integer so that

$$\lim_{t \to 1} (1 - t)^d P_M(t) < \infty$$

**Proposition 11.5.** $d(M) \leq s$ for all f.g. graded $A$-modules $M$. 


Proof.

\[
\frac{1 - t}{1 - t^k} = \frac{1}{1 + t + \cdots + t^{k-1}} \to \frac{1}{k}
\]
as \(t \to 1\). So, \((1 - t)^sP_M(t)\) converges as \(t \to 1\). \(\square\)

Lemma 11.6. If \(x \in A_k\) is \(M\)-regular then \(d(M) = d(M/xM) + 1\).

Proposition 11.7. If \(A\) has an \(M\)-regular sequence of length \(r\) then
\(d(M) \geq r\)

In Example 11.3, \(d = s = r\). So, \(d(K[x_1, \cdots, x_d]) = d\). The idea is that \(d(M)\) is the number of generators of \(A\) which act as independent variables on \(M\).

11.1.4. Hilbert function. The Hilbert function gives a formula for \(\ell(M_n)\) for large \(n\).

Theorem 11.8. Suppose that \(k_i = 1\) for \(i = 1, 2, \cdots, s\). Then \(\ell(M_n)\) is a polynomial of degree \(d(M) - 1\) for large \(n\).

Proof. By Theorem 11.1 we have:

\[
\sum \ell(M_n)t^n = \frac{f(t)}{(1 - t)^s}
\]
where \(f(t)\) is a polynomial. By definition of \(d = d(M)\) we have \(f(t) = (1 - t)^{s-d}g(t)\) where \(g(1) \neq 0\). So,

\[
\sum \ell(M_n)t^n = \frac{g(t)}{(1 - t)^d} = \sum \binom{n + d - 1}{d - 1}t^n g(t)
\]

Suppose that \(g(t) = \sum_{k=0}^{m} a_k t^k = a_0 + a_1 t + \cdots + a_m t^m\). Then

\[
\sum \ell(M_n)t^n = \sum_{k=0}^{m} \sum_{n+k} \binom{n + d - 1}{d - 1}t^n g(t)
\]

When \(n \geq m\) we get

\[
\ell(M_n) = \sum_{k=0}^{m} a_k \binom{n - k + d - 1}{d - 1}
\]
(When \(n < m\) we take the sum over \(0 \leq k \leq n\).) Since the binomial coefficient is a polynomial in \(n\) of degree \(d - 1\) with leading coefficient \(1/(d - 1)!\) regardless of the value of \(k\), this formula for \(\ell(M_n)\) is a polynomial in \(n\) of degree \(d - 1\) with leading coefficient

\[
\frac{1}{(d - 1)!} \sum_{k=0}^{m} a_k = \frac{g(1)}{(d - 1)!} \neq 0
\]

\(\square\)
Definition 11.9. The **Hilbert function** of $M$ is the polynomial function
\[ h_M(n) = \ell(M_n) \]
for sufficiently large $n$.

11.1.5. **Samuel function.** For each $n$ consider the finite sum:
\[ g(n) = \ell(M_0) + \ell(M_1) + \cdots + \ell(M_{n-1}). \]

**Lemma 11.10.** $g(n)$ is a polynomial of degree $d(M)$ for sufficiently large $n$.

**Proof.** For $m > m_0$ we have
\[ \ell(M_m) = h_M(m) = c_0 + c_1 m + \cdots + c_{d-1} m^{d-1} \]
Therefore
\[ g(n) = \sum_{m=0}^{n-1} h_M(m) + \text{const} \]
for $n > m_0$. But, anyone who has taught Calculus knows that $\sum_{m=0}^{n-1} m^k$ is a polynomial in $n$ of degree $k + 1$. The lemma follows. \hfill $\square$

Suppose that $A$ is a Noetherian local ring with unique maximal ideal $\mathfrak{m}$ and quotient field $K = A/\mathfrak{m}$. Let $M$ be any f.g. $A$-module. For any $n$, $M/\mathfrak{m}^nM$ is a f.g. module over the Artin local ring $A/\mathfrak{m}^n$. Therefore it has a finite length:
\[ \ell(M/\mathfrak{m}^nM) = \dim_K(M/\mathfrak{m}M \oplus \mathfrak{m}M/\mathfrak{m}^2M \oplus \cdots \oplus \mathfrak{m}^{n-1}M/\mathfrak{m}^nM) \]

**Proposition 11.11.** For sufficiently large $n$, $\ell(M/\mathfrak{m}^nM)$ is a polynomial in $n$ of degree $\leq s = \dim_K(\mathfrak{m}/\mathfrak{m}^2)$.

**Proof.** Take the associated graded ring
\[ Gr(A) = A/\mathfrak{m} \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \mathfrak{m}^2/\mathfrak{m}^3 \oplus \cdots \]
has positive ideal $Gr(A)_+ = \mathfrak{m}/\mathfrak{m}^2 \oplus \mathfrak{m}^2/\mathfrak{m}^3 \oplus \cdots$ generated by $s$ elements of degree 1. The associated $Gr(A)$-module
\[ Gr(M) = M/\mathfrak{m}M \oplus \mathfrak{m}M/\mathfrak{m}^2M \oplus \cdots \]
is a f.g. $Gr(A)$-module with $d(Gr(M)) \leq s$ by Proposition 11.5. So, $\ell(M/\mathfrak{m}^nM)$ is a polynomial of degree $d(Gr(M)) \leq s$ for sufficiently large $n$ by the lemma above. \hfill $\square$

**Definition 11.12.** The **Samuel function** for $M$ is defined to be the polynomial function given for large $n$ by:
\[ \chi^M_{\mathfrak{m}}(n) = \ell(M/\mathfrak{m}^nM) \]
\[ \deg \chi^M_{\mathfrak{m}}(n) = d(Gr(M)) \leq \dim_K(\mathfrak{m}/\mathfrak{m}^2) \]
11.2. Noetherian local rings. Suppose that $A$ is a Noetherian local ring with maximal ideal $\mathfrak{m}$. We will prove the main theorem of dimension theory which says:

$$d(A) = \text{dim } A = \delta(A)$$

**Definition 11.13.** If $A$ is a Noetherian ring $\delta(A)$ is defined to be the smallest number of elements which can generate a proper ideal $q$ so that $A/q$ is Artinian. When $A$ is Noetherian local, this is equivalent to saying that $q$ is $\mathfrak{m}$-primary. If $M$ is a f.g. $A$-module we define $\delta(M)$ to be $\delta(A/\text{Ann}(M))$.

Once we review the definition of $d(A)$ we will see that we have already proven that

$$d(A) \leq \delta(A)$$

11.2.1. definition of $d(M)$. Suppose that $s = \delta(A)$ and $q$ is an $\mathfrak{m}$-primary ideal generated by $s$ elements. Then

$$Gr_q(A) = A/q \oplus q/q^2 \oplus q^2/q^3 \oplus \cdots$$

is a graded Noetherian local ring whose positive ideal $Gr_q(A)_+ = q/q^2 \oplus q^2/q^3 \oplus \cdots$ is generated by $s$ homogeneous elements in degree 1. Suppose that $M$ is a f.g. $A$-module and $(M_n)$ is a stable $q$-filtration. Then

$$Gr(M) = M/M_1 \oplus M_1/M_2 \oplus M_2/M_3 \oplus \cdots$$

is a f.g. $Gr_q(A)$-module. This implies that the Poincaré series

$$P_M(t) = \sum_{n=0}^{\infty} \ell(M_n/M_{n+1})t^n = \frac{f(t)}{(1-t)^s}$$

where $f(t) \in \mathbb{Z}[t]$. If we reduce the fraction we get $(1-t)^d$ in the denominator where $d = d(M)$. So, clearly $d \leq s = \delta(A)$. Replacing $A$ with $A/\text{Ann } M$ we get

$$d(M) \leq \delta(M)$$

(in particular $d(A) \leq \delta(A)$) with one particular definition of $d(M)$.

Next recall that

$$h(n) = \ell(M_n/M_{n+1})$$

is a polynomial in $n$ of degree $d(M) - 1$ for large $n$. This was called the Hilbert function of $M$. As a formal consequence of this theorem we concluded that, for large $n$,

$$g(n) = \ell(M/M_n)$$
is a polynomial in \( n \) of degree \( d(M) \). (The theorem was stated only in the case \( M_n = m^n M \) but the same proof works for any filtration.) In the particular case \( M_n = q^n M \) we have the Samuel function
\[
\chi^M_q(n) = \ell(M/q^n M)
\]

11.2.2. \( d(M) \) is well defined. We will show that \( d(M) \) is independent of the choice of \( q \) and the choice of the filtration \( (M_n) \).

**Proposition 11.14.** Given any \( m \)-primary ideal \( q \) and any stable \( q \)-filtration \( (M_n) \), the polynomials \( g(n) = \ell(M/M_n) \) and \( \chi^M_q(n) = \ell(M/q^n M) \) have the same degree and leading coefficient. Furthermore, this leading coefficient is a positive rational number.

**Proof.** \((M_n)\) being \( q \)-stable means \( \exists n_0 \) s.t.
\[
q^{n+n_0} M \subseteq M_{n+n_0} = q^n M_{n_0} \subseteq q^n M
\]
This implies that
\[
\ell(M/q^{n+n_0}) \geq \ell(M/M_{n+n_0}) \geq \ell(M/q^n M)
\]
\[
\chi^M_q(n + n_0) \geq g(n + n_0) \geq \chi^M_q(n)
\]
But \( \chi^M_q(n + n_0) \) and \( \chi^M_q(n) \) are polynomials in \( n \) of the same degree and the same leading coefficient. So, \( g(n + n_0) = \ell(M/M_{n+n_0}) \) and therefore \( g(n) \) also has the same degree and leading coefficient. Positivity is obvious. \( \Box \)

**Proposition 11.15.** The degree of the polynomial \( \chi^M_q(n) \) is independent of the choice of the \( m \)-primary ideal \( q \).

**Proof.** If \( q \) is \( m \)-primary then \( m^a \subseteq q \subseteq m \) for some \( a \). This implies \( m^{na} \subseteq q^n \subseteq m^n \) which implies
\[
\ell(M/m^{na} M) \geq \ell(M/q^n M) \geq \ell(M/m^n M)
\]
\[
\chi^M_m(na) \geq \chi^M_q(n) \geq \chi^M_m(n)
\]
But it is easy to see that \( \chi^M_m(na) \) is a polynomial in \( n \) with the same degree, say \( d \), as \( \chi^M_m(n) \) (with leading coefficient multiplied by \( a^d \)). So, \( \chi^M_q(n) \) is a polynomial of degree \( d \) in \( n \). \( \Box \)

**Theorem 11.16.** If \( A \) is a Noetherian local ring and \( M \) is a finitely generated \( A \)-module then \( d(M) \leq \delta(M) \). In particular
\[
d(A) \leq \delta(A).
\]
11.2.3. **Krull dimension.** The plan is to prove that

\[ \delta(A) \geq d(A) \geq \dim A \geq \delta(A) \]

showing that all three are equal. We discussed the definitions of the first two terms and proved the first inequality. Now we prove the second. The proof will be by induction on \( d(A) \) (we have not yet proven that \( \dim A \) is finite).

**Lemma 11.17.** If \( 0 \to M' \to M \to M'' \to 0 \) is a short exact sequence of \( A \)-modules then

\[ d(M) = \max(d(M'), d(M'')) \]

Furthermore, if \( M \cong M' \) then \( d(M) > d(M'') \).

**Proof.** For each \( n \) we have the short exact sequence:

\[ 0 \to M' \cap m^n M \to m^n M \to m^n M'' \to 0 \]

By the Artin-Rees Lemma, \((M' \cap m^n M)\) is a stable \( m \)-filtration of \( M' \). This exact sequence gives an exact sequence of quotient modules

\[ 0 \to \frac{M'}{M' \cap m^n M} \to \frac{M}{m^n M} \to \frac{M''}{m^n M''} \to 0 \]

Since length is additive, this gives

\[ \ell\left( \frac{M}{m^n M} \right) = \ell\left( \frac{M'}{M' \cap m^n M} \right) + \ell\left( \frac{M''}{m^n M''} \right) \]

\[ \chi_m^M(n) = g(n) + \chi_m^{M''}(n) \]

By Proposition 11.14, the polynomial \( g(n) \) has degree \( d(M') \) and \( \chi_m^{M''}(n) \) has degree \( d(M'') \). Since both of these polynomials have positive leading terms, the leading terms cannot cancel and their sum has degree equal to the maximum of the two degrees.

In the special case when \( M \cong M' \), we use the fact that \( g(n) \) will have the same degree and the same leading coefficient as \( \chi_m^M(n) \) causing \( \chi_m^{M''}(n) \) to have smaller degree.

**Theorem 11.18.** If \( A \) is a Noetherian local ring with Krull dimension \( \dim A \) then

\[ d(A) \geq \dim A \]

**Proof.** By induction on \( d(A) \). If \( d(A) = 0 \) then \( m^n = 0 \) for some \( n \) making \( A \) Artinian with \( \dim A = 0 \). So, suppose \( d = d(A) > 0 \) and the theorem holds for \( B \) with \( d(B) < d \).
We are trying to show that \( \dim A \leq d \). So, suppose by contradiction that \( \dim A > d \). Then there exists a tower of prime ideals in \( A \) of length \( d + 1 \):

\[
p_0 \subsetneq p_1 \subsetneq p_2 \subsetneq \cdots \subsetneq p_{d+1}
\]

Choose any \( x \in p_1 - p_0 \). Let \( \overline{x} \) be the image of \( x \) in the integral domain \( A' = A/p_0 \). Then \( \overline{x} \) is \( A' \)-regular (not a unit and not a zero divisor).

Therefore we have an exact sequence of \( A \)-modules

\[
0 \to A/p_0 \to A/p_0 \to A'/\overline{x} \to 0
\]

By the lemma,

\[
d = d(A) \geq d(A') > d(A'/\overline{x}) \geq d(A/p_1)
\]

By induction on \( d \) we have

\[
d(A/p_1) \geq \dim A/p_1 \geq d
\]

since \( p_1 \subsetneq p_2 \subsetneq \cdots \subsetneq p_{d+1} \) gives a tower of prime ideals of length \( d \) in \( A/p_1 \). This is a contradiction proving the theorem.

11.2.4. the fundamental theorem of dimension theory.

**Theorem 11.19.** If \( A \) is a Noetherian local ring then

\[
\dim A \geq \delta(A)
\]

**Proof.** Let \( d = \dim A \).

**Claim 1** \( \exists x_1, \cdots, x_d \in m \) so that

\[
\dim A/(x_1, \cdots, x_i) \leq d - i
\]

In particular, \( \dim A/(x_1, \cdots, x_d) = 0 \iff (x_1, \cdots, x_d) \) is \( m \)-primary \( \Rightarrow \) \( \delta(A) \leq d \). So, it suffices to prove this claim.

We prove Claim 1 by induction on \( d \). If \( d = 0 \) the statement is clearly true. So, suppose that \( d > 0 \).

**Claim 2** If \( d = \dim A > 0 \) then \( \exists x_1 \in m \) so that

\[
\dim(A/(x_1)) < d
\]

By induction on \( i \), Claim 2 implies Claim 1. So, it suffices to prove Claim 2.

Since \( A \) is Noetherian, \( A \) contains only finitely many minimal primes. (They are the minimal primes associated to the primary ideals in the primary decomposition of \( 0 \).) Let \( p_1, \cdots, p_i \) be these minimal primes. Then

\[
m \supseteq \bigcup p_i
\]

(If \( m = \bigcup p_i \) then by prime avoidance we would have \( m \subseteq p_i \) for some \( i \) contradicting the assumption that \( \dim A > 0 \).)
Let \( x_1 \in \mathfrak{m} \) so that \( x \notin \mathfrak{p}_i \) for any \( i \). Then \( \dim(A/(x_1)) < d \) because, if not, there would be a tower of prime ideals in \( A/(x_1) \) of length \( d \). This would give a tower of prime ideals

\[
q_0 \subsetneq q_1 \subsetneq \cdots \subsetneq q_d
\]

in \( A \) all of which contain \((x_1)\). But \( x_1 \) is not contained in any minimal prime. So, \( q_0 \) is not minimal. So, the tower can be extended contradicting the hypothesis that \( \dim A = d \).

This proves Claim 2 which implies Claim 1 which proves the theorem.

\[ \square \]

**Corollary 11.20** (Fundamental theorem of dimension theory). *If \( A \) is a Noetherian local ring then*

\[
\dim A = d(A) = \delta(A)
\]

*In particular, \( \dim A \) is finite.*

**Corollary 11.21.** If \( A \) is a Noetherian local ring with maximal ideal \( \mathfrak{m} \) and \( k = A/\mathfrak{m} \) then

\[
\dim A \leq \dim_k \mathfrak{m}/\mathfrak{m}^2
\]

**Corollary 11.22.** In any Noetherian ring, every prime ideal has finite height. In particular, the prime ideals satisfy the DCC.
11.3. **regular local rings.** Suppose that $A$ is a Noetherian local ring of dimension $d$. Then there exist $d$ elements $x_1, \ldots, x_d$ in $A$ which generate an $m$-primary ideal $q$ in $A$. Such a collection of elements is called a **system of parameters** for $A$.

**Definition 11.23.** A system of parameters for $A$ is called **regular** if it generates the maximal ideal $m$. If $A$ has a regular system of parameters it is called a **regular local ring**.

11.3.1. **first characterization of regular local rings.**

**Proposition 11.24.** Suppose that $A$ is a Noetherian local ring with $\dim A = d$. Then the following are equivalent.

1. $Gr_m(A) \cong k[t_1, \ldots, t_d]$.
2. $\dim_m m/m^2 = d$.
3. $m$ is generated by $d$ elements.
4. $A$ is a regular local ring.

**Proof.** Certainly (1) $\Rightarrow$ (2) and (3) $\Leftrightarrow$ (4) by definition of regular local ring. Nakayama’s lemma implies that (2) $\Leftrightarrow$ (3). To show that (3) $\Rightarrow$ (1) suppose that $x_1, \ldots, x_d$ generate $m$, i.e., they form a regular system of parameters for $A$. Then we get an epimorphism of graded rings

$$\varphi : k[t] = k[t_1, \ldots, t_d] \twoheadrightarrow Gr_m(A)$$

sending $t_i$ to the image $\overline{x}_i$ of $x_i$ in $m/m^2$. If this homomorphism is not an isomorphism then it has a nonzero element in its kernel, say $f(t)$. But $k[t]$ is an integral domain. So, $f(t)$ is not a zero divisor. And it cannot be a unit, being in the kernel of $\varphi$. So $f$ is regular and thus

$$d(k[t_1, \ldots, t_d]/(f)) = d(k[t_1, \ldots, t_d]) - 1 = d - 1$$

But $Gr_m(A)$ is a quotient of $k[t_1, \ldots, t_d]/(f)$. So,

$$d = d(Gr_m(A)) \leq d(k[t_1, \ldots, t_d]/(f)) = d - 1$$

which is a contradiction. So, $\varphi$ must be an isomorphism as claimed. \qed

**Corollary 11.25.** A regular local ring is an integral domain.

**Proof.** By Krull’s theorem 10.26, $\bigcap m^n = 0$. Therefore the following lemma applies. \qed

**Lemma 11.26.** Suppose that $a$ is an ideal in $A$ so that $\bigcap a^n = 0$. Suppose that $Gr_a(A)$ is an integral domain. Then $A$ is an integral domain.
Proof. Suppose that \(x, y\) are nonzero elements of \(A\). Since \(\bigcap a^n = 0\), \(x \in a^n\) for some \(n \geq 0\) and similarly \(y \in a^m\) for some \(m \geq 0\). Then the images of these elements \(\bar{x} \in a^n/a^{n+1}\), \(\bar{y} \in a^m/a^{m+1}\) are nonzero in \(\text{Gr}_a(A)\) and therefore their product \(\bar{x} \bar{y} \in a^{n+m}/a^{n+m+1}\) is nonzero (since \(\text{Gr}_a(A)\) is an integral domain). This implies that \(xy \notin a^{n+m+1}\) and in particular, \(xy \neq 0\). \(\square\)

**Corollary 11.27.** A ring \(A\) is a regular local ring of dimension 1 iff it is a discrete valuation ring.

**Proposition 11.28.** Let \(A\) be a Noetherian local ring. Then \(\hat{A}\) is a Noetherian local ring with the same dimension and \(A\) is regular iff \(\hat{A}\) is regular.

11.3.2. **Koszul complex.** We will use the Koszul complex to prove part of the following well known theorem.

**Theorem 11.29 (Serre).** Suppose that \(A\) is a Noetherian local ring. Then the following are equivalent.

1. \(A\) is regular.
2. \(\text{gl} \dim A < \infty\)
3. \(\text{gl} \dim A = \dim A\).

Suppose that \(x_1, \ldots, x_d \in A\). Then the **Koszul complex** \(K_*(x_1, \ldots, x_d)\) is the chain complex of f.g. free \(A\) modules given by:

\[
K_n(x) = \text{free } A\text{-module generated by } e_{i_1 i_2 \cdots i_n} \cong A_{(n)}^d
\]

where \(1 \leq i_1 < i_2 < \cdots < i_n \leq d\) with differential \(\partial : K_n \to K_{n-1}\) given by

\[
\partial(e_{i_1 i_2 \cdots i_n}) = \sum_{j=1}^{n} (-1)^{j+1} x_{i_j} e_{i_1 \cdots \hat{i_j} \cdots i_n}
\]

It is easy to see that \(\partial \partial = 0\). Also \(0 \leq n \leq d\):

\[
0 \to K_d(x) \overset{\partial_d}{\to} K_{d-1}(x) \to \cdots \to K_1(x) \overset{\partial_1}{\to} K_0(x)
\]

Claim: \(K_0(x) = A\) and the image of \(\partial_1 : K_1(x) \to A = K_0(x)\) is the ideal \(q = (x_1, \ldots, x_d)\). So the Koszul complex is augmented over \(A/q\):

\[
0 \to K_d(x) \overset{\partial}{\to} K_{d-1}(x) \to \cdots \to K_1(x) \overset{\partial}{\to} K_0(x) \overset{\epsilon}{\to} A/q \to 0
\]

**Lemma 11.30.** If \(A\) is a regular local ring of \(\dim A = d\) and \(x_1, \ldots, x_d\) is a regular system of parameters for \(A\) then the Koszul complex \(K_*(x)\) is a free resolution of \(k = A/m\).
We went over the basic definitions in class. The *global dimension* of a ring is the supremum of the projective dimensions of all f.g. modules. This is defined in terms of projective modules.

**Definition 11.31.** A module $P$ is **projective** if it satisfies the property that for any epimorphism of modules $p : X \to Y$ and any homomorphism $f : P \to Y$ there exists a morphism $\tilde{f} : P \to X$ so that $f = p \circ \tilde{f}$.

$$
\begin{array}{ccc}
X & \xrightarrow{p} & Y \\
\downarrow{\exists \tilde{f}} & & \\
P & \xrightarrow{f} & Y 
\end{array}
$$

If we take $X = A^n$ and $Y = P$ with $f = \text{id}_P$, we get the following characterization of projective modules.

**Proposition 11.32.** A module $P$ is projective iff it is isomorphic to a direct summand of a free module. Every f.g. projective module is isomorphic to a direct summand of $A^n$ for some positive integer $n$.

**Definition 11.33.** The **projective dimension** $\text{pd} \dim M$ of any $A$-module $M$ is defined to be the smallest integer $n$ so that there exists an exact sequence:

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \xrightarrow{\epsilon} M \to 0$$

where the $P_i$ are all projective modules.

In class I “rotated” the augmentation map $\epsilon : P_0 \to M$ to get the following diagram:

$$
P_* : \quad \begin{array}{ccc}
0 & \longrightarrow & P_n & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & 0 \\
& & & & & & & & \downarrow{\epsilon} & & & & \\
M : & \cdots & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0
\end{array}
$$

The exactness of the original sequence is equivalent to the statement that $\epsilon$ is a *quasi-isomorphism* of chain complexes.

**Definition 11.34.** A *quasi-isomorphism* of chain complexes is defined to be a chain map which induces an isomorphism in homology in all degrees.

In the case at hand, the projective chain complex $P_*$ is exact except in degree 0 where the map $P_1 \to P_0$ is not onto. So, $H_0(P_*) \cong M$ and $\epsilon : P_0 \to M$ gives a quasi-isomorphism $P_* \simeq M$.

**Definition 11.35.** The **global dimension** of any ring $A$ is defined to be the supremum of $\text{pd} \dim M$ for all f.g. $A$-modules $M$. 

Lemma 11.30 (proved below) implies that \( \text{pr dim}_A k \leq d \). We need to strengthen this to an equality and then we need another lemma:

**Lemma 11.36.** If \( A \) is a local ring with maximal ideal \( \mathfrak{m} \) and \( k = A/\mathfrak{m} \) then

\[
gl \dim A = \text{pr dim } k.
\]

**Proof.** By definition of global dimension we have \( gl \dim A \geq \text{pr dim } k \).

So, we need to show the reverse inequality. Suppose that \( \text{pr dim } M = n \). Then there is a projective complex

\[
0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow 0
\]

which is quasi-isomorphic to \( M \). \( Tor_n^A(M, k) \) is the kernel of the mapping

\[
P_n \otimes k \rightarrow P_{n-1} \otimes k
\]

But this mapping is 0 by Nakayama (assuming the projective complex \( P_* \) is minimal). So, \( Tor_n^A(M, k) \neq 0 \). But

\[
Tor_n^A(k, M) \cong Tor_n^A(M, k)
\]

So, this implies that \( \text{pr dim } k \geq n \). Therefore,

\[
\text{pr dim } M \leq \text{pr dim } k
\]

for all f.g. \( A \)-modules \( M \). The lemma follows. \( \square \)

We need another lemma to do the induction step for the proof that the Koszul complex is exact.

**Lemma 11.37.** If \( x_1, \cdots, x_d \) is a regular system of parameters for a regular local ring \( A \) then \( A/(x_1, \cdots, x_n) \) is a regular local ring of dimension \( d - n \) for \( n = 1, 2, \cdots, d \).

**Proof.** By induction it suffices to do the case \( n = 1 \). In that case, \( x_1 \) is a regular element since \( A \) is an integral domain. So, \( \dim A/(x_1) = d - 1 \).

But the images \( \overline{x}_2, \cdots, \overline{x}_d \) of \( x_2, \cdots, x_d \) generate the maximal ideal of \( A/(x_1) \). So, \( A/(x_1) \) is regular. \( \square \)

**Proof of Lemma 11.30.** The proof is by induction on the length of the regular sequence. The induction hypothesis is:

Claim: For all \( 1 \leq n \leq d \), the Koszul complex \( K_s(x_1, \cdots, x_n) \) is quasi-isomorphic to \( A/(x_1, \cdots, x_n) \).

For \( n = d \) this is the statement that we want to prove.

Start with \( n = 1 \). Since \( x_1 \) is regular, multiplication by \( x_1 \) is a monomorphism and we have an exact sequence:

\[
0 \rightarrow A \xrightarrow{x_1} A \rightarrow A/(x_1) \rightarrow 0
\]
When we “rotate” the augmentation map this becomes a quasi-isomorphism $K_\ast(x_1) \simeq A/(x_1)$. So, the Claim holds for $n = 1$.

The case $n = 2$ shows the idea of the induction step. In this case the Koszul complex $K_\ast(x_1, x_2)$ looks like:

\[
\begin{array}{c}
\xrightarrow{-x_2} A e_1 \xrightarrow{x_1} A \\
\oplus \xrightarrow{x_1} A e_2 \xrightarrow{x_2} A
\end{array}
\]

This complex contains two copies of the smaller complex $K_\ast(x_1)$ one is shifted and the boundary map from one copy to the other is multiplication by $x_2$. This means we have a short exact sequence of chain complexes:

\[
0 \to K_\ast(x_1) \to K_\ast(x_1, x_2) \to \Sigma K_\ast(x_1) \to 0
\]

We know by induction that $K_\ast(x_1) \simeq A/(x_1)$. So, we can compare this to the complex:

\[
0 \to A/(x_1) \to X_\ast \to \Sigma A/(x_1) \to 0
\]

where $X_\ast$ is the chain complex:

\[
X_\ast : A/(x_1) \xrightarrow{x_2} A/(x_1)
\]

This is the Koszul complex $K_\ast(x_2)$ for the regular local ring $A/(x_1)$. So, it is quasi-isomorphic to $A/(x_1, x_2)$. We have the following commuting diagram where the rows are exact and the first and third vertical maps are quasi-isomorphisms by induction. So, the middle vertical arrow is also a quasi-isomorphism by the 5-lemma.

\[
\begin{array}{c}
0 \to K_\ast(x_1) \to K_\ast(x_1, x_2) \to \Sigma K_\ast(x_1) \to 0 \\
\simeq \downarrow \simeq \downarrow \simeq \\
0 \to A/(x_1) \to X_\ast \to \Sigma A/(x_1) \to 0
\end{array}
\]

Therefore

\[
K_\ast(x_1, x_2) \simeq X_\ast \simeq A/(x_1, x_2).
\]

The next steps are similar. \hfill \Box

11.3.3. multiplicity. I gave a very brief summary of another use for the Koszul complex. Namely, in the equation for the leading terms of the Samuel function

\[
\chi_\mathfrak{q}^A(n) = \ell(A/\mathfrak{q}^n)
\]
Theorem 11.38. Suppose that $A$ is a Noetherian local ring of dimension $d$ and $x_1, \cdots, x_d$ is a system of parameters for $A$. Let $\mathfrak{q} = (x_1, \cdots, x_d)$. Then the leading term of the Samuel function $\chi^A(\mathfrak{q})$ is equal to
\[
\frac{\chi(K_*(x_1, \cdots, x_d))}{d!} t^d
\]
where $\chi(K_*(x))$ is the Euler characteristic of the Koszul complex:
\[
\chi(K_*(x)) = \sum_{i=0}^{d} (-1)^i \ell(H_i(K_*(x)))
\]
For example in the case of a regular local ring the leading term is
\[
\frac{1}{d!} t^d
\]
and
\[
H_i(K(x)) = \begin{cases} 
  k & \text{if } i = 0 \\
  0 & \text{otherwise}
\end{cases}
\]