MATH 223A NOTES 2011
LIE ALGEBRAS

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Date: September 6, 2011.

1. Basic Concepts

1.1. Definition of Lie algebra.

Definition 1.1.1. By an *(nonassociative) algebra* over a field F we mean a vector space A together with an F-bilinear operation $A \times A \to A$ which is usually written $(x, y) \mapsto xy$.

The adjective "nonassociative" means "not necessarily associative". An *associative algebra* is an algebra A whose multiplication rule is associative: $x(yz) = (xy)z$ for all $x, y, z \in A$. The existence of a unit 1 is not assumed.

Definition 1.1.2. Let L be a vector space over a field F. Then a bilinear operation $[\cdot\cdot] : L \times L \to L$ sending (x, y) to $[xy]$ is called a *bracket* if it satisfies the following two conditions.

[L1] $[xx] = 0$ for all $x \in L$.
[L2] (*Jacobi identity*) $[x[yz]] + [y[zx]] + [z[xy]] = 0$ for all $x, y, z \in L$.

A vector space L with a bracket $[\cdot\cdot]$ is called a *Lie algebra*. This is an example of a nonassociative algebra.

Let us analyze the two conditions. Condition [L1] implies:

[L1′] $[xy] = -[yx]$ for all $x, y \in L$. (Bracket is *skew commutative*.)
Proof: $[(x + y)(x + y)] = 0 = [xx] + [xy] + [yx] + [yy] = [xy] + [yx]$.

Conversely, if the characteristic of the field F is not equal to 2 then [L1′] implies that $2[xx] = 0$ implies [L1]. So, [L1] and [L1′] are equivalent when *char* $F \neq 2$.

The second condition [L2] can be rewritten as follows:

$$[x[yz]] = [[xy]z] + [[zx]y].$$

The term $[[zx]y]$ prevents L from being associative. Since z, x, y are arbitrary we obtain:

Proposition 1.1.3. *A Lie algebra is associative if and only if* $[[LL]L] = 0$.

The notation $[[LL]L]$ indicates the vector subspace of L generated by all expressions $[[xy]z]$.

Definition 1.1.4. A *(Lie) subalgebra* of a Lie algebra L is defined to be a vector subspace K so that $[KK] \subseteq K$.

For example, $[LL]$ is always a Lie subalgebra of L.

Definition 1.1.5. A *homomorphism* of Lie algebras is a linear map $\varphi : L \to L'$ so that $\varphi([xy]) = [\varphi(x)\varphi(y)]$ for all $x, y \in L$.

1.2. Examples.

Example 1.2.1. The simplest example of a Lie algebra is given by letting $[xy] = 0$ for all $x, y \in L$ where L is any vector space over F. All conditions are clearly satisfied. A Lie algebra satisfying this condition (usually written as $[LL] = 0$) is called *abelian*.

The word "abelian" comes from one standard interpretation of the bracket. Suppose that A is an associative algebra over F. Then the *commutator* $[xy]$ is defined by $[xy] = xy - yx$. This is easily seen to be a bracket and is also called the *Lie bracket* of the associative algebra.

Example 1.2.2. Suppose that V is any vector space over F. We define $\mathfrak{gl}(V)$ to be the Lie algebra of all F-linear endomorphisms of V under the Lie bracket operation. A Lie subalgebra of $\mathfrak{gl}(V)$ is called a *linear Lie algebra*.

Definition 1.2.3. A *representation* of the Lie algebra L is defined to be a Lie algebra homomorphism $L \to \mathfrak{gl}(V)$ for some vector space V. The representation is called *faithful* if this homomorphism is injective: $L \hookrightarrow \mathfrak{gl}(V)$.

1.2.1. *linear Lie algebras.* There is a well-known theorem (due to Ado in characteristic 0 and Iwasawa in characteristic p) what every finite dimensional Lie algebra has a faithful finite dimensional representation. I.e., it is isomorphic to a linear Lie algebra. So, our finite dimensional examples are all linear. What are the finite dimensional linear Lie algebras?

If $V = F^n$ then $\mathfrak{gl}(V)$ is denoted $\mathfrak{gl}(n, F)$. This is the vector space of all $n \times n$ matrices with coefficients in F with Lie bracket given by commutator: $[xy] = xy - yx$. A subalgebra is given by a subset of $\mathfrak{gl}(n, F)$ which is closed under this bracket and under addition and scalar multiplication.

Example 1.2.4. Let $\mathfrak{sl}(n, F) \subseteq \mathfrak{gl}(n, F)$ denote the set of all $n \times n$ matrices with trace equal to zero.

(1) $\mathrm{Tr}([xy]) = \mathrm{Tr}(xy) - \mathrm{Tr}(yx) = 0$. So, $\mathfrak{sl}(n, F)$ is closed under $[\,]$.
(2) $\mathrm{Tr}(x + y) = \mathrm{Tr}(x) + \mathrm{Tr}(y) = 0$.
(3) $\mathrm{Tr}(ax) = a\,\mathrm{Tr}(x) = 0$

Therefore, $\mathfrak{sl}(n, F)$ is a linear Lie algebra.

Proposition 1.2.5. *Suppose that $f : V \times V \to F$ is a bilinear form. Then the set of all $x \in \mathfrak{gl}(V)$ so that*

$$f(x(v), w) + f(v, x(w)) = 0$$

for all $v, w \in V$ is a Lie subalgebra of $\mathfrak{gl}(V)$ which we denote $\mathfrak{o}(V, f)$

Proof. It is clear that $\mathfrak{o}(V, f)$ is a vector subspace since the defining equation is linear in x. The following calculation shows that it is closed under Lie bracket.

$$f(xy(v), w) + f(y(v), x(w)) = 0$$

$$f(yx(v), w) + f(x(v), y(w)) = 0$$

$$f(v, xy(w)) + f(x(v), y(w)) = 0$$

$$f(v, yx(w)) + f(y(v), x(w)) = 0$$

If we take the alternating sum (+ - + -) of these equations we see that

$$f([xy](v), w) + f(v, [xy](w)) = 0$$

□

Example 1.2.6. Particular examples of the above definition are as follows.

(1) Suppose that f is a nondegenerate symmetric bilinear form on V. Then $\mathfrak{o}(V, f)$ is called the *orthogonal Lie algebra* relative to f.
(2) Suppose that f is a nondegenerate skew symmetric form on V: $f(v, v) = 0$ for all $v \in V$. (If *char* $F \neq 2$ this is equivalent to the condition that $f(v, w) = -f(w, v)$ for all v, w.) In this case $\dim V = 2n$ (even) and $\mathfrak{o}(V, f)$ is called the *symplectic Lie algebra* relative to f.

We will look at these examples in more detail later.

Example 1.2.7. Other easy examples of linear Lie algebras are:

(1) $\mathfrak{t}(n, F) \subseteq \mathfrak{gl}(n, F)$, the set of upper triangular $n \times n$ matrices over F
(2) $\mathfrak{n}(n, F) \subseteq \mathfrak{t}(n, F)$, the set of strictly upper triangular matrices (with 0 on the diagonal).
(3) $\mathfrak{d}(n, F) \subseteq \mathfrak{t}(n, F)$, the set of diagonal $n \times n$ matrices with coefficients in F.

1.3. Derivations.

Definition 1.3.1. Suppose that A is a nonassociative algebra over F. Then a *derivation* on A is a linear function $\delta : A \to A$ so that

$$\delta(xy) = \delta(x)y + x\delta(y)$$

for all $x, y \in A$. The set of all derivations on A is denoted $Der(A)$.

Proposition 1.3.2. *$Der(A)$ is a Lie subalgebra of $\mathfrak{gl}(A)$.*

Proof. □

Go back to the definition of a Lie algebra. Using the skew symmetry condition [L1′], Condition [L2] can be rephrased as:

$$[z[xy]] = [[zx]y] + [x[zy]$$

In other words, the bracket by z operation $\mathrm{ad}_z(\cdot) = [z(\cdot)]$ satisfies:

$$\mathrm{ad}_z[xy] = [\mathrm{ad}_z(x)y] + [x\mathrm{ad}_z(y)]$$

So any Lie algebra *acts on itself by derivations*. This gives a homomorphism:

$$\mathrm{ad} : L \to Der(A)$$

called the *adjoint representation*.

1.4. Abstract Lie algebras.

We could simply start with the definition and try to construct all possible Lie algebras. Take $L = F^n$.

$n = 1$: Show that all one dimensional Lie algebras are abelian.
$n = 2$: If $L = F^2$ there are, up to isomorphism, exactly two examples.
$n = 3$: Example: Take $L = \mathbb{R}^3$ and take the Lie bracket to be the *cross product*.

$$[xy] = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

Verify the Jacobi identiy. Which Lie algebra is this?