2. Ideals and homomorphisms

2.1. Ideals.

Definition 2.1.1. An *ideal* in a Lie algebra L is a vector subspace I so that $[LI] \subseteq I$. In other words, $[ax] \in I$ for all $a \in L, x \in I$.

Example 2.1.2. (1) 0 and L are always ideals in L.
(2) If L is abelian then every vector subspace is an ideal.
(3) $[LL]$ is an ideal in L called the *derived algebra* of L.

Proposition 2.1.3. *Every 2-dimensional Lie algebra contains a 1-dimensional ideal.*

Proof. As we saw last time, there are only two examples of a 2-dimensional Lie algebra: Either the basis elements commute, in which case L is abelian, or they don't commute, in which case $[LL]$ is 1-dimensional. In both cases, L has a 1-dimensional ideal. □

Proposition 2.1.4. *The kernel of an homomorphism of Lie algebras $\varphi : L \to L'$ is an ideal in L. (The image is a subalgebra of L'.) Conversely, for any ideal $I \subseteq L$, L/I is a Lie algebra and I is the kernel of the quotient map $L \to L/I$.*

Proof. If $x \in \ker\varphi$ and $a \in L$ then $\varphi[ax] = [\varphi(a)\varphi(x)] = [\varphi(a)0] = 0$. So, $[ax] \in \ker\varphi$. Conversely, if $I \subset L$ is an ideal, $a \in L, x \in I$ then

$$[(a+I)(x+I)] \subseteq [ax] + [aI] + [Ix] + [II] \subseteq [ax] + I$$

So, the bracket is well-defined in L/I, $[\varphi(a)\varphi(x)] = \varphi[ax]$ and $I = \ker\varphi$. □

Proposition 2.1.5. *Any epimorphism (surjective homomorphism) of Lie algebras $\varphi : L \to L'$ gives a 1-1 correspondence between ideals I' in L' and ideals I of L containing* $\ker\varphi$.

Proof. The correspondence is given by $I' = \varphi(I)$. This is an ideal since $[L', I'] = [\varphi(L), \varphi(I)] = \varphi[LI] \subseteq \varphi(I) = I'$ and $I = \varphi^{-1}(I')$ which is an ideal since it is the kernel of $L \to L' \to L'/I'$. □

Example 2.1.6. The *center* of a Lie algebra L is defined to be

$$Z(L) = \{x \in L \mid [xL] = 0\}.$$

Then $Z(L)$ is an ideal in L. $Z(L)$ also the kernel of the adjoint representation $\mathrm{ad} : L \to \mathfrak{gl}(L)$. If $I \subseteq L$ is an ideal, then the adjoint representation restricts to a representation on I: $\mathrm{ad} : L \to \mathfrak{gl}(I)$ and the kernel of this is the *centralizer* $Z_L(I)$ of I in L. This is an ideal in L since it is a kernel.

Example 2.1.7. Consider the Lie algebra $\mathfrak{n}(3, F)$. This is 3-dimensional with basis x_{12}, x_{23}, x_{13} and the only nontrivial bracket is $[x_{12}x_{23}] = x_{13}$. Then $L = \mathfrak{n}(3, F)$ has the property that $[LL] = Z(L)$ is one-dimensional. Conversely, it is easy to see that any 3-dimensional Lie algebra with $[LL] = Z(L)$ must be isomorphic to $\mathfrak{n}(3, F)$.

Example 2.1.8. $\mathrm{Tr} : \mathfrak{gl}(n, F) \to F$ is a homomorphism of Lie algebras (with F the abelian Lie algebra) since $\mathrm{Tr}[xy] = \mathrm{Tr}(xy) - \mathrm{Tr}(yx) = 0 = [\mathrm{Tr}(x), \mathrm{Tr}(y)]$. Therefore, $\mathfrak{sl}(n, F)$ is an ideal in $\mathfrak{gl}(n, F)$.

Exercise 2.1.9. What is the center of $\mathfrak{sl}(2,F)$?

A Lie algebra is called *simple* if it is nonabelian and has no nontrivial proper ideals.

Theorem 2.1.10. *$\mathfrak{sl}(2,F)$ is simple if char $F \neq 2$.*

$\mathfrak{sl}(2,F)$ is a key example of a Lie algebra which you need to understand very thoroughly.

Proof. Since $L = \mathfrak{sl}(2,F)$ is 3-dimensional we just need to show that it has no ideals of dimension 1 and no ideals of dimension 2.

The standard basis for $\mathfrak{sl}(2,F)$ is given by

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and computation gives:

$$[hx] = 2x, \quad [hy] = -2y, \quad [xy] = h$$

This proves that $[LL] = L$. This implies that L has no 2-dimensional ideal I since then L/I would be 1-dimensional and thus abelian which would imply that $[LL] \subseteq I$.

This also implies that $L = \mathfrak{sl}(2,F)$ has no 1-dimensional ideal I since, in that case, L/I would be 2-dimensional and thus would have a 1-dimensional ideal by Proposition 2.1.3 and this would correspond to a 2-dimensional ideal in L by Proposition 2.1.5. Therefore, $\mathfrak{sl}(n,F)$ is simple. □

Proposition 2.1.11. *Every simple Lie algebra is linear.*

Proof. The center of L is trivial. So, the adjoint representation gives an embedding $\mathrm{ad} : L \hookrightarrow \mathfrak{gl}(L)$. □

2.2. **nilpotent elements and automorphisms.** First, suppose that *char* $F = 0$ and δ is a nilpotent derivation of any nonassociative algebra A. I.e., $\delta^k = 0$ for some k. Then we claim that

$$\exp \delta = \sum_{i=0}^{k-1} \frac{\delta^i}{i!}$$

is an automorphism of A. It is easy to see that this is a linear automorphism of A since it has the form $1 + \eta$ where η is nilpotent. So, the inverse is $1 - \eta + \eta^2 - \cdots$ which is a finite sum. The following lemma shows that $\exp(-\delta)$ is the inverse of $\exp \delta$.

Lemma 2.2.1. *Suppose that char $F = 0$ and f, g are commuting nilpotent endomorphism of some vector space V (or, more generally, commuting elements of any associative algebra with unity) then* $\exp(f+g) = \exp(f)\exp(g)$. *In particular,* $\exp(f)$ *is a linear automorphism of V with inverse* $\exp(-f)$.

Proof. Since f, g commute we have:

$$(f+g)^n = \sum_{i=0}^{n} \binom{n}{i} f^i g^{n-i}$$

Since $char\, F = 0$ we can divide both sides by $n!$ to get

$$\frac{(f+g)^n}{n!} = \sum_{i+j=n} \frac{f^i}{i!}\frac{g^j}{j!}$$

Note that for sufficiently large n, all terms are zero since f, g are nilpotent. Thus we can sum over all $n \geq 0$ to get

$$\exp(f+g) = \exp(f)\exp(g)$$

as claimed. □

To see that $\exp\delta$ is a homomorphism of algebras, we can use the following trick:

$$\delta(xy) = \delta \circ \mu(x,y)$$

where $\mu : A \times A \to A$ is multiplication. Then $\delta \circ \mu = \mu \circ (\delta_1 + \delta_2)$ where

$$\delta_1(x,y) = (\delta x, y), \quad \delta_2(x,y) = (x, \delta y)$$

Since δ_1, δ_2 commute, we can use the lemma to get:

$$\exp\delta(xy) = \exp\delta \circ \mu(x,y) = \mu \circ \exp(\delta_1 + \delta_2)(x,y) = \mu \circ \exp(\delta_1)\exp(\delta_2)(x,y)$$

$$= \exp\delta(x)\exp\delta(y)$$

Definition 2.2.2. Suppose that x is an element of a Lie algebra L so that ad_x is nilpotent. Then $\exp\mathrm{ad}_x$ is called an *inner automorphism* of L.

Proposition 2.2.3. *Suppose that L is the Lie algebra of an associative algebra A with unity* 1*. Let $x \in A$ be a nilpotent element. Then:*

1. $\exp x = \sum x^k/i!$ *is a unit in A.*
2. ad_x *is a nilpotent endomorphism of L.*
3. $\exp\mathrm{ad}_x$ *is conjugation by* $\exp x$*.*

Proof. By the Lemma, $\exp(-x)$ is the inverse of $\exp x$. The other statement follow from the following trick: Write $\mathrm{ad}_x = \lambda_x + \rho_{-x}$ where λ_x is "left multiplication by x" and ρ_{-x} is "right multiplication by $-x$": $\lambda_x(y) = xy$ and $\rho_{-x}(y) = -yx$. Since left and right multiplication are commuting operations,

$$\exp\mathrm{ad}_x = \exp(\lambda_x + \rho_{-x}) = \exp\lambda_x \exp\rho_{-x}$$

But clearly, $\exp\lambda_x = \lambda_{\exp x}$ and $\exp\rho_{-x} = \rho_{\exp(-x)}$. So,

$$\exp\mathrm{ad}_x(y) = (\exp x)y(\exp(-x)) = (\exp x)y(\exp(x)^{-1})$$

□

2.3. **Exercises.** What about derivations in characteristic p?

(1) Show that δ^p is a derivation.
(2) For the Lie algebra of an associative algebra over a field of characteristic p show that $\mathrm{ad}_x^p = \mathrm{ad}_{x^p}$.
(3) (in any characteristic) Let $\varphi : V \times V \to F$ be a nondegenerate skew-symmetric bilinear pairing. Then the *Heisenberg algebra* of f is given by $L = V \oplus F$ with $[(v, a)(w, b)] = (0, f(v, w))$. Show that $[LL] = Z(L)$ is 1-dimensional and that all Lie algebras with this property are Heisenberg algebras.
(4) If I, J are ideals in L then show that $I \cap J$ and $[IJ]$ are ideals. How are these related? Give an example where these are different.
(5) The *normalizer* $N_L(K)$ of a subalgebra $K \subseteq L$ is the set of all $x \in L$ so that $[xK] \subseteq K$. Note that K is an ideal In $N_L(K)$. Show that $\delta(n, F)$ and $\mathfrak{t}(n, F)$ are self-normalizing in $\mathfrak{gl}(n, F)$.