2. Ideals and homomorphisms

2.1. Ideals.

Definition 2.1.1. An ideal in a Lie algebra $L$ is a vector subspace $I$ so that $[LI] \subseteq I$. In other words, $[ax] \in I$ for all $a \in L, x \in I$.

Example 2.1.2. (1) $0$ and $L$ are always ideals in $L$.
(2) If $L$ is abelian then every vector subspace is an ideal.
(3) $[LL]$ is an ideal in $L$ called the derived algebra of $L$.

Proposition 2.1.3. Every 2-dimensional Lie algebra contains a 1-dimensional ideal.

Proof. As we saw last time, there are only two examples of a 2-dimensional Lie algebra: Either the basis elements commute, in which case $L$ is abelian, or they don’t commute, in which case $[LL]$ is 1-dimensional. In both cases, $L$ has a 1-dimensional ideal. □

Proposition 2.1.4. The kernel of an homomorphism of Lie algebras $\varphi : L \rightarrow L'$ is an ideal in $L$. (The image is a subalgebra of $L'$.) Conversely, for any ideal $I \subseteq L$, $L/I$ is a Lie algebra and $I$ is the kernel of the quotient map $L \rightarrow L/I$.

Proof. If $x \in \ker \varphi$ and $a \in L$ then $\varphi(ax) = [\varphi(a)\varphi(x)] = [\varphi(a)0] = 0$. So, $[ax] \in \ker \varphi$. Conversely, if $I \subseteq L$ is an ideal, $a \in L, x \in I$ then

$$[(a+I)(x+I)] \subseteq [ax] + [aI] + [Ix] + [II] \subseteq [ax] + I$$

So, the bracket is well-defined in $L/I$, $[\varphi(a)\varphi(x)] = \varphi(ax)$ and $I = \ker \varphi$. □

Proposition 2.1.5. Any epimorphism (surjective homomorphism) of Lie algebras $\varphi : L \rightarrow L'$ gives a 1-1 correspondence between ideals $I'$ in $L'$ and ideals $I$ of $L$ containing $\ker \varphi$.

Proof. The correspondence is given by $I' = \varphi(I)$. This is an ideal since $[L', I'] = [\varphi(L), \varphi(I)] = \varphi[LI] \subseteq \varphi(I) = I'$ and $I = \varphi^{-1}(I')$ which is an ideal since it is the kernel of $L \rightarrow L' \rightarrow L'/I'$.

Example 2.1.6. The center of a Lie algebra $L$ is defined to be

$$Z(L) = \{x \in L \mid [xL] = 0\}.$$ 

Then $Z(L)$ is an ideal in $L$. $Z(L)$ also the kernel of the adjoint representation $\text{ad} : L \rightarrow \mathfrak{gl}(L)$. If $I \subseteq L$ is an ideal, then the adjoint representation restricts to a representation on $I$: $\text{ad} : L \rightarrow \mathfrak{gl}(I)$ and the kernel of this is the centralizer $Z_L(I)$ of $I$ in $L$. This is an ideal in $L$ since it is a kernel.

Example 2.1.7. Consider the Lie algebra $\mathfrak{n}(3, F)$. This is 3-dimensional with basis $x_{12}, x_{23}, x_{13}$ and the only nontrivial bracket is $[x_{12}x_{23}] = x_{13}$. Then $L = \mathfrak{n}(3, F)$ has the property that $[LL] = Z(L)$ is one-dimensional. Conversely, it is easy to see that any 3-dimensional Lie algebra with $[LL] = Z(L)$ must be isomorphic to $\mathfrak{n}(3, F)$.

Example 2.1.8. $\text{Tr} : \mathfrak{gl}(n, F) \rightarrow F$ is a homomorphism of Lie algebras (with $F$ the abelian Lie algebra) since $\text{Tr}[xy] = \text{Tr}(xy) - \text{Tr}(yx) = 0 = [\text{Tr}(x), \text{Tr}(y)]$. Therefore, $\mathfrak{sl}(n, F)$ is an ideal in $\mathfrak{gl}(n, F)$.
**Exercise 2.1.9.** What is the center of \( \mathfrak{sl}(2, F) \)?

A Lie algebra is called *simple* if it is nonabelian and has no nontrivial proper ideals.

**Theorem 2.1.10.** \( \mathfrak{sl}(2, F) \) is simple if \( \text{char } F \neq 2 \).

\( \mathfrak{sl}(2, F) \) is a key example of a Lie algebra which you need to understand very thoroughly.

**Proof.** Since \( L = \mathfrak{sl}(2, F) \) is 3-dimensional we just need to show that it has no ideals of dimension 1 and no ideals of dimension 2.

The standard basis for \( \mathfrak{sl}(2, F) \) is given by

\[
\begin{align*}
x &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\
y &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\
h &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{align*}
\]

and computation gives:

\[
\begin{align*}
[hx] &= 2x, \\
[hy] &= -2y, \\
[xy] &= h
\end{align*}
\]

This proves that \([LL] = L\). This implies that \( L \) has no 2-dimensional ideal \( I \) since then \( L/I \) would be 1-dimensional and thus abelian which would imply that \([LL] \subseteq I\).

This also implies that \( L = \mathfrak{sl}(2, F) \) has no 1-dimensional ideal \( I \) since, in that case, \( L/I \) would be 2-dimensional and thus would have a 1-dimensional ideal by Proposition 2.1.3 and this would correspond to a 2-dimensional ideal in \( L \) by Proposition 2.1.5. Therefore, \( \mathfrak{sl}(2, F) \) is simple. □

**Proposition 2.1.11.** Every simple Lie algebra is linear.

**Proof.** The center of \( L \) is trivial. So, the adjoint representation gives an embedding \( \text{ad} : L \hookrightarrow \mathfrak{gl}(L) \). □

**2.2. nilpotent elements and automorphisms.** First, suppose that \( \text{char } F = 0 \) and \( \delta \) is a nilpotent derivation of any nonassociative algebra \( A \). I.e., \( \delta^k = 0 \) for some \( k \). Then we claim that

\[
\exp \delta = \sum_{i=0}^{k-1} \frac{\delta^i}{i!}
\]

is an automorphism of \( A \). It is easy to see that this is a linear automorphism of \( A \) since it has the form \( 1 + \eta \) where \( \eta \) is nilpotent. So, the inverse is \( 1 - \eta + \eta^2 - \cdots \) which is a finite sum. The following lemma shows that \( \exp(-\delta) \) is the inverse of \( \exp(\delta) \).

**Lemma 2.2.1.** Suppose that \( \text{char } F = 0 \) and \( f, g \) are commuting nilpotent endomorphism of some vector space \( V \) (or, more generally, commuting elements of any associative algebra with unity) then \( \exp(f + g) = \exp(f)\exp(g) \). In particular, \( \exp(f) \) is a linear automorphism of \( V \) with inverse \( \exp(-f) \).

**Proof.** Since \( f, g \) commute we have:

\[
(f + g)^n = \sum_{i=0}^{n} \binom{n}{i} f^i g^{n-i}
\]
Since char $F = 0$ we can divide both sides by $n!$ to get

$$\frac{(f + g)^n}{n!} = \sum_{i+j=n} \frac{f^i g^j}{i! j!}$$

Note that for sufficiently large $n$, all terms are zero since $f, g$ are nilpotent. Thus we can sum over all $n \geq 0$ to get

$$\exp(f + g) = \exp(f)\exp(g)$$
as claimed.

To see that $\exp \delta$ is a homomorphism of algebras, we can use the following trick:

$$\delta(xy) = \delta \circ \mu(x, y)$$

where $\mu : A \times A \to A$ is multiplication. Then $\delta \circ \mu = \mu \circ (\delta_1 + \delta_2)$ where

$$\delta_1(x, y) = (\delta x, y), \quad \delta_2(x, y) = (x, \delta y)$$

Since $\delta_1, \delta_2$ commute, we can use the lemma to get:

$$\exp \delta(xy) = \exp \delta \circ \mu(x, y) = \mu \circ \exp(\delta_1 + \delta_2)(x, y) = \mu \circ \exp(\delta_1)\exp(\delta_2)(x, y)$$

$$= \exp \delta(x) \exp \delta(y)$$

**Definition 2.2.2.** Suppose that $x$ is an element of a Lie algebra $L$ so that $\text{ad}_x$ is nilpotent. Then $\exp \text{ad}_x$ is called an *inner automorphism* of $L$.

**Proposition 2.2.3.** Suppose that $L$ is the Lie algebra of an associative algebra $A$ with unity 1. Let $x \in A$ be a nilpotent element. Then:

1. $\exp x = \sum x^k/k!$ is a unit in $A$.
2. $\text{ad}_x$ is a nilpotent endomorphism of $L$.
3. $\exp \text{ad}_x$ is conjugation by $\exp x$.

**Proof.** By the Lemma, $\exp(-x)$ is the inverse of $\exp x$. The other statement follow from the following trick: Write $\text{ad}_x = \lambda_x + \rho_{-x}$ where $\lambda_x$ is “left multiplication by $x$” and $\rho_{-x}$ is “right multiplication by $-x$”: $\lambda_x(y) = xy$ and $\rho_{-x}(y) = -yx$. Since left and right multiplication are commuting operations,

$$\exp \text{ad}_x = \exp (\lambda_x + \rho_{-x}) = \exp \lambda_x \exp \rho_{-x}$$

But clearly, $\exp \lambda_x = \lambda_{\exp x}$ and $\exp \rho_{-x} = \rho_{\exp(-x)}$. So,

$$\exp \text{ad}_x(y) = (\exp x)y(\exp(-x)) = (\exp x)y(\exp(x)^{-1})$$

$\square$
2.3. **Exercises.** What about derivations in characteristic $p$?

1. Show that $\delta^p$ is a derivation.
2. For the Lie algebra of an associative algebra over a field of characteristic $p$ show that $\text{ad}_{\delta^p} = \text{ad}_{\varphi^p}$.
3. (in any characteristic) Let $\varphi : V \times V \to F$ be a nondegenerate skew-symmetric bilinear pairing. Then the *Heisenberg algebra* of $f$ is given by $L = V \oplus F$ with $[(v, a)(w, b)] = (0, f(v, w))$. Show that $[LL] = Z(L)$ is $1$-dimensional and that all Lie algebras with this property are Heisenberg algebras.

4. If $I, J$ are ideals in $L$ then show that $I \cap J$ and $[IJ]$ are ideals. How are these related? Give an example where these are different.

5. The *normalizer* $N_L(K)$ of a subalgebra $K \subseteq L$ is the set of all $x \in L$ so that $[xK] \subseteq K$. Note that $K$ is an ideal In $N_L(K)$. Show that $\delta(n, F)$ and $t(n, F)$ are self-normalizing in $\mathfrak{gl}(n, F)$.