3. Nilpotent and solvable Lie algebras

I can’t find my book. The following is from Fulton and Harris.

**Definition 3.0.1.** A Lie algebra is **solvable** if its iterated derived algebra is zero. In other words, $D^kL = 0$ where $DL = [LL], D^2L = [(DL)(DL)] = [[LL][LL]],$ etc. This is a recursive definition: The $k$-th derived algebra of $L$ is the $k - 1$st derived algebra of $[LL].$

**Definition 3.0.2.** A Lie algebra is **nilpotent** of class $k$ if

$$\text{ad}^k_L(L) = [L \cdots [L[L[L]] \cdots ] = 0$$

for some $k$.

Note that if $L$ is nilpotent of class $k$ then $\text{ad}^k_x = 0$ for all $x \in L$ since $\text{ad}^k_x(y) = [x[x \cdots [xy] \cdots ]] \in [L[L[L \cdots [LL] \cdots ]]] = 0.$ Every element of $L$ is ad-nilpotent.

3.1. **Engel’s Theorem.** This is converse of the above statement.

**Theorem 3.1.1** (Engel). If $L$ is a finite dimensional Lie algebra in which every element is ad-nilpotent then $L$ is nilpotent.

We prove this theorem in a sequence of lemmas. The first lemma allows us to assume that $L$ is a linear Lie algebra.

**Lemma 3.1.2.** Suppose the image of the adjoint representation $\text{ad} : L \to \mathfrak{gl}(L)$ is a nilpotent subalgebra of $\mathfrak{gl}(L)$ of class $k$ then $L$ is nilpotent of class $\leq k + 1.$

**Proof.** The kernel of the adjoint representation is the center $Z = Z(L).$ If $L/Z$ is nilpotent of class $k$ then $\text{ad}^k_{L/Z}(L/Z) = \text{ad}^k_L(L) \subseteq Z.$ But $[LZ] = 0$ so $\text{ad}^{k+1}_L(L) = [[L[L[L[L \cdots [LL] \cdots ]]]] \subseteq [LZ] = 0.$ \hfill $\square$

If every element of $L$ is ad-nilpotent then the image of the adjoint representation consists of nilpotent endomorphism of $L$ considered as a vector space.

**Lemma 3.1.3.** Suppose that $L$ is a subalgebra of $\mathfrak{gl}(V)$ where $V$ is a nonzero finite dimensional vector space over $F.$ Suppose that every element of $L$ is a nilpotent endomorphism of $V.$ Then there exists a nonzero element $v \in V$ so that $x(v) = 0$ for all $x \in L.$

Some people call this Engle’s Theorem since it is the key step in the proof of the theorem. To see that the lemma implies the theorem, let $K$ be the subspace of $V$ spanned by the vector $v.$ Then the action of $L$ on $V$ induces an action on $V/K$ which is nilpotent. So, the image of $L$ in $\mathfrak{gl}(V/K)$ is a nilpotent Lie algebra of class, say $k.$ This implies that $L^k(V) \subseteq K.$ So, $L^{k+1}(V) = 0$ making $L$ nilpotent of class $k + 1.$

**Proof of key lemma.** The proof is by induction on the dimension of $L.$ If $L$ is one-dimensional, the lemma is clear. So, suppose $\dim L \geq 2.$ Let $J$ be a maximal proper subalgebra of $L.$

**Claim 1:** $J$ is an ideal of codimension 1 in $L.$


**Remark**

Since \( J \) is smaller than \( L \), there is a nonzero vector \( v \in V \) so that \( J(v) = 0 \). Let \( W \) be the set of all \( v \in V \) so that \( J(v) = 0 \). Then \( W \) is a nonzero vector subspace of \( V \). Take \( y \in L, y \notin J \).

**Claim 2:** \( y(W) \subseteq W \).

**Pf:** We need to show that, for any \( x \in J \) and \( w \in W \), \( xy(w) = 0 \). This follows from the following calculation:

\[ xy(w) = [xy](w) + yx(w) = 0 + 0 = 0 \]

since \([xy] \in J\) and \( J(w) = 0 \).

Since \( y \) is nilpotent and sends \( W \) into \( W \), there is some nonzero \( w \in W \) so that \( y(w) = 0 \). Since \( J(w) = 0 \), we conclude that \( L(w) = 0 \) proving the lemma.

**Exercise 3.1.4.**

1. Show that \( n(n, F) \) is nilpotent of class \( n - 1 \). Hint: there is a filtration of \( V = F^n \) by vector subspaces

\[
0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n
\]

so that \( x(V_i) \subseteq V_{i-1} \) for all \( x \in n(n, F) \). Such a sequence of subspaces of \( V \) is called a flag.

2. Show that, for any linear Lie algebra \( L \subseteq \mathfrak{gl}(V) \), the existence of a flag in \( V \) with the property that \( x(V_i) \subseteq V_{i-1} \) implies that \( L \) is a subalgebra of \( n(n, F) \) up to isomorphism. (Assume \( \dim V = n \) is finite.)

3. Show that any nilpotent subalgebra of \( \mathfrak{gl}(V) \) is isomorphic to a subalgebra of \( n(n, F) \) where \( n = \dim V \).

**3.2. Lie’s theorem.**

When we go to solvable Lie algebras we need the ground field \( F \) to be algebraically closed of characteristic 0. So, we might as well assume that \( F = \mathbb{C} \).

Then we have the following theorem whose statement and proof are similar to the statement and proof of the key lemma for Engel’s Theorem.

**Theorem 3.2.1 (Lie).** Suppose that \( L \subseteq \mathfrak{gl}(V) \) is a solvable linear Lie algebra over \( \mathbb{C} \). Then there exists a nonzero vector \( v \in V \) which is a simultaneous eigenvector of every element of \( L \), i.e., \( x(v) = \lambda(x)v \) for every \( x \in L \) where \( \lambda(x) \in \mathbb{C} \).

**Remark 3.2.2.** Before proving this we note that, the function \( \lambda : L \to \mathbb{C} \) is a linear map.

1. \( ax(v) = a\lambda(x)v \). So, \( \lambda(ax) = a\lambda(x) \).
2. \( (x + y)(v) = x(v) + y(v) = \lambda(x)v + \lambda(y)v = (\lambda(x) + \lambda(y))(v) \). So, \( \lambda(x + y) = \lambda(x) + \lambda(y) \).
Furthermore, note that if $\lambda : L \to \mathbb{C}$ is a linear map then the equation $x(v) = \lambda(x)(v)$ is a linear equation in $x$. Therefore, if this equation holds for all $x$ in a spanning set for $L$ then it holds for all $x$ in $L$.

**Proof.** The proof is by induction on the dimension of $L$. If $L$ is one dimensional then we are dealing with one endomorphism $x$ of $V$. Since $\mathbb{C}$ is algebraically closed, the characteristic polynomial of $x$ has a root $\lambda \in \mathbb{C}$ and a corresponding eigenvector $v$. $(x(v) = \lambda v$. So, $\lambda_x = \lambda$.) So, suppose that $\dim L \geq 2$.

The next step is to find a codimension one ideal $J$ in $L$. Since $[LL] \subseteq L$, this is easy. Take any codimension one vector subspace of $L/[LL]$ and let $J$ be the inverse image of this in $L$.

By induction there is a nonzero vector $v \in V$ and a linear map $\lambda : J \to \mathbb{C}$ so that $x(v) = \lambda(x)v$ for all $x \in J$. Let $W$ be the set of all $v \in V$ with this property (for the same linear function $\lambda$). Let $y \in L, y \notin J$.

**Claim:** $y(W) \subseteq W$.

Suppose for a moment that this is true. Then, we can find an eigenvalue $\lambda(y)$ and eigenvector $w \in W$ so that $y(w) = \lambda(y)w$. By the remark, this extended linear map $\lambda$ satisfies the desired equation. The Claim is proved more generally in the following lemma.

**Lemma 3.2.3.** Suppose that $J$ is an ideal in $L$, $V$ is a representation of $L$ and $\lambda : J \to F$ is a linear map. Assume $\text{char } F = 0$. Let

$$W = \{v \in V \mid x(v) = \lambda(x)v \forall x \in J\}$$

Then $y(W) \subseteq W$ for all $y \in L$.

**Proof.** Define $W_i$ recursively as follows. $W_0 = 0$ and

$$W_{k+1} = \{v \in V \mid x(v) = \lambda(x)(v) \in W_k \forall x \in J\}$$

Then $W = W_1 \subseteq W_2 \subseteq \cdots$ since $V$ is finite dimensional we must have $W_k = W_{k+1}$ for some $k$. This means that $(x - \lambda(x))^k = 0$ on $W_k$. So, for any $x \in J$, the matrix of $x$ as an endomorphism of $W_k$ is upper triangular with $\lambda(x)$ on the diagonal and we have

$$\text{Tr}(x|W_k) = \lambda(x) \dim W_k$$

**Claim** For any $y \in L, y(W_k) \subseteq W_{k+1} = W_k$.

Suppose for a moment that this is true. Then, for any $x \in J, y \in L$ we have $\text{Tr}([xy]|W_k) = 0 = \lambda[xy] \dim W_k$. Since $\text{char } F = 0$ this implies $\lambda[xy] = 0$. We can now show that $y(w) \in W$ for all $y \in L, w \in W$:

$$x(y(w)) = yx(w) + [xy](w) = \lambda(x)y(w) + \lambda[xy](w) = \lambda(x)y(w)$$

Thus it suffices to prove the claim.

Proof of Claim: $y(W_i) \subseteq W_{i+1}$. To prove this we must show that, for any $w \in W_i, x \in J$ we have $(x - \lambda(x))y(w) \in W_i$. This is a calculation similar to the one above:

$$x(y(w)) - \lambda(x)y(w) = y((x - \lambda(x))(w) + [xy](w) \in y(W_{i-1}) + W_i \subseteq W_i$$

proving the claim by induction. (It is clear when $i = 0$.)
Exercise 3.2.4. (1) Show that $t(n, F)$ is solvable.

(2) Show that any solvable subalgebra of $\mathfrak{gl}(n, \mathbb{C})$ is, up to isomorphism, isomorphic to a subalgebra of $t(n, \mathbb{C})$.

(3) If $J$ is an ideal in $L$ then show that $L$ is solvable if and only if $J$ and $L/J$ are solvable.

(4) Prove that the following are equivalent.
   (a) $L$ is solvable.
   (b) $L$ has a sequence of ideals $L \supset J_1 \supset J_2 \cdots \supset J_n = 0$ so that $J_k/J_{k+1}$ is abelian for each $k$.
   (c) $L$ has a sequence of subalgebras $L \supset L_1 \supset L_2 \cdots \supset L_n = 0$ so that $L_{k+1}$ is an ideal in $L_k$ and $L_k/L_{k+1}$ is abelian for all $k$. 