3. Nilpotent and solvable Lie algebras

I can't find my book. The following is from Fulton and Harris.

Definition 3.0.1. A Lie algebra is *solvable* if its iterated derived algebra is zero. In other words, $D^kL = 0$ where $DL = [LL], D^2L = [(DL)(DL)] = [[LL][LL]]$, etc. This is a recursive definition: The *k-th derived algebra* of L is the $k-1$st derived algebra of $[LL]$.

Definition 3.0.2. A Lie algebra is *nilpotent* of class k if

$$\mathrm{ad}_L^k(L) = \underbrace{[L\cdots[L[L[L}_{k}\, L]]]\cdots] = 0$$

for some k.

Note that if L is nilpotent of class k then $\mathrm{ad}_x^k = 0$ for all $x \in L$ since $\mathrm{ad}_x^k(y) = [x[x[x\cdots[xy]\cdots]]] \in [L[L[L\cdots[LL]\cdots]]] = 0$. Every element of L is ad-nilpotent.

3.1. Engel's Theorem.

This is converse of the above statement.

Theorem 3.1.1 (Engel)**.** *If L is a finite dimensional Lie algebra in which every element is ad-nilpotent then L is nilpotent.*

We prove this theorem in a sequence of lemmas. The first lemma allows us to assume that L is a linear Lie algebra.

Lemma 3.1.2. *Suppose the image of the adjoint representation* $\mathrm{ad} : L \to \mathfrak{gl}(L)$ *is a nilpotent subalgebra of $\mathfrak{gl}(L)$ of class k then L is nilpotent of class $\leq k+1$.*

Proof. The kernel of the adjoint representation is the center $Z = Z(L)$. If L/Z is nilpotent of class k then $\mathrm{ad}_{L/Z}^k(L/Z) =$ is the same as $\mathrm{ad}_L^k(L) \subseteq Z$. But $[LZ] = 0$ so $\mathrm{ad}_L^{k+1}(L) = [L\mathrm{ad}_L^k(L)] \subseteq [LZ] = 0$. □

If every element of L is ad-nilpotent then the image of the adjoint representation consists of nilpotent endomorphism of L considered as a vector space.

Lemma 3.1.3. *Suppose that L is a subalgebra of $\mathfrak{gl}(V)$ where V is a nonzero finite dimensional vector space over F. Suppose that every element of L is a nilpotent endomorphism of V. Then there exists a nonzero element $v \in V$ so that $x(v) = 0$ for all $x \in L$.*

Some people call this Engle's Theorem since it is the key step in the proof of the theorem. To see that the lemma implies the theorem, let K be the subspace of V spanned by the vector v. Then the action of L on V induces an action on V/K which is nilpotent. So, the image of L in $\mathfrak{gl}(V/K)$ is a nilpotent Lie algebra of class, say k. This implies that $L^k(V) \subseteq K$. So, $L^{k+1}(V) = 0$ making L nilpotent of class $k+1$.

Proof of key lemma. The proof is by induction on the dimension of L. If L is one-dimensional, the lemma is clear. So, suppose $\dim L \geq 2$. Let J be a maximal proper subalgebra of L.

<u>Claim 1</u>: J is an ideal of codimension 1 in L.

Pf: J acts, by the adjoint action on L and this action leaves J invariant. Therefore, J act on the quotient L/J. By induction on dimension, there is a nonzero vector $y+J \in L/J$ so that ad_J kills this vector. In other words, $[Jy] \subseteq J$. This implies that J and y span a subalgebra of L with dimension one more than the dimension of J. By maximality of J, this subalgebra is equal to L. So, J has codimension 1. Also, J is an ideal since $[LJ] \subseteq [JJ] + [yJ] \subseteq J$.

Since J is smaller than L, there is a nonzero vector $v \in V$ so that $J(v) = 0$. Let W be the set of all $v \in V$ so that $J(v) = 0$. Then W is a nonzero vector subspace of V. Take $y \in L, y \notin J$.

Claim 2: $y(W) \subseteq W$.

Pf: We need to show that, for any $x \in J$ and $w \in W$, $xy(w) = 0$. This follows from the following calculation:

$$xy(w) = [xy](w) + yx(w) = 0 + 0 = 0$$

since $[xy] \in J$ and $J(w) = 0$.

Since y is nilpotent and sends W into W, there is some nonzero $w \in W$ so that $y(w) = 0$. Since $J(w) = 0$, we conclude that $L(w) = 0$ proving the lemma. □

Exercise 3.1.4.
(1) Show that $\mathfrak{n}(n, F)$ is nilpotent of class $n - 1$. Hint: there is a filtration of $V = F^n$ by vector subspaces

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots V_n$$

so that $x(V_i) \subseteq V_{i-1}$ for all $x \in \mathfrak{n}(n, F)$. Such a sequence of subspaces of V is called a *flag*.
(2) Show that, for any linear Lie algebra $L \subseteq \mathfrak{gl}(V)$, the existence of a flag in V with the property that $x(V_i) \subseteq V_{i-1}$ implies that L is a subalgebra of $\mathfrak{n}(n, F)$ up to isomorphism. (Assume $\dim V = n$ is finite.)
(3) Show that any nilpotent subalgebra of $\mathfrak{gl}(V)$ is isomorphic to a subalgebra of $\mathfrak{n}(n, F)$ where $n = \dim V$.

3.2. **Lie's theorem.** When we go to solvable Lie algebras we need the ground field F to be algebraically closed of characteristic 0. So, we might as well assume that $F = \mathbb{C}$.

Then we have the following theorem whose statement and proof are similar to the statement and proof of the key lemma for Engel's Theorem.

Theorem 3.2.1 (Lie)**.** *Suppose that $L \subseteq \mathfrak{gl}(V)$ is a solvable linear Lie algebra over $\mathbb{C}$. Then there exists a nonzero vector $v \in V$ which is a simultaneous eigenvector of every element of L, i.e., $x(v) = \lambda(x)v$ for every $x \in L$ where $\lambda(x) \in \mathbb{C}$.*

Remark 3.2.2. Before proving this we note that, the function $\lambda : L \to \mathbb{C}$ is a linear map.

(1) $ax(v) = a\lambda(x)v$. So, $\lambda(ax) = a\lambda(x)$.
(2) $(x + y)(v) = x(v) + y(v) = \lambda(x)v + \lambda(y)v = (\lambda(x) + \lambda(y))(v)$. So, $\lambda(x + y) = \lambda(x) + \lambda(y)$.

Furthermore, note that if $\lambda : L \to \mathbb{C}$ is a linear map then the equation $x(v) = \lambda(x)(v)$ is a linear equation in x. Therefore, if this equation holds for all x in a spanning set for L then it holds for all x in L.

Proof. The proof is by induction on the dimension of L. If L is one dimensional then we are dealing with one endomorphism x of V. Since $\mathbb{C}$ is algebraically closed, the characteristic polynomial of x has a root $\lambda \in \mathbb{C}$ and a corresponding eigenvector v. ($x(v) = \lambda v$. So, $\lambda_x = \lambda$.) So, suppose that $\dim L \geq 2$.

The next step is to find a codimension one ideal J in L. Since $[LL] \subsetneq L$, this is easy. Take any codimension one vector subspace of $L/[LL]$ and let J be the inverse image of this in L.

By induction there is a nonzero vector $v \in V$ and a linear map $\lambda : J \to \mathbb{C}$ so that $x(v) = \lambda(x)v$ for all $x \in J$. Let W be the set of all $v \in V$ with this property (for the same linear function λ). Let $y \in L, y \notin J$.

<u>Claim</u>: $y(W) \subseteq W$.

Suppose for a moment that this is true. Then, we can find an eigenvalue $\lambda(y)$ and eigenvector $w \in W$ so that $y(w) = \lambda(y)w$. By the remark, this extended linear map λ satisfies the desired equation. The Claim is proved more generally in the following lemma. $\square$

Lemma 3.2.3. *Suppose that J is an ideal in L, V is a representation of L and $\lambda : J \to F$ is a linear map. Assume char $F = 0$. Let*

$$W = \{v \in V \mid x(v) = \lambda(x)v \ \forall x \in J\}$$

Then $y(W) \subseteq W$ for all $y \in L$.

Proof. Define W_i recursively as follows. $W_0 = 0$ and

$$W_{k+1} = \{v \in V \mid x(v) - \lambda(x)(v) \in W_k \ \forall x \in J\}$$

Then $W = W_1 \subseteq W_2 \subseteq \cdots$ since V is finite dimensional we must have $W_k = W_{k+1}$ for some k. This means that $(x - \lambda(x))^k = 0$ on W_k. So, for any $x \in J$, the matrix of x as an endomorphism of W_k is upper triangular with $\lambda(x)$ on the diagonal and we have

$$\mathrm{Tr}(x|W_k) = \lambda(x) \dim W_k$$

<u>Claim</u> For any $y \in L, y(W_k) \subseteq W_{k+1} = W_k$.

Suppose for a moment that this is true. Then, for any $x \in J, y \in L$ we have $\mathrm{Tr}([xy]|W_k) = 0 = \lambda[xy] \dim W_k$. Since *char* $F = 0$ this implies $\lambda[xy] = 0$. We can now show that $y(w) \in W$ for all $y \in L, w \in W$:

$$x(y(w)) = yx(w) + [xy](w) = \lambda(x)y(w) + \lambda[xy](w) = \lambda(x)y(w)$$

Thus it suffices to prove the claim.

Proof of Claim: $y(W_i) \subseteq W_{i+1}$. To prove this we must show that, for any $w \in W_i, x \in J$ we have $(x - \lambda(x))y(w) \in W_i$. This is a calculation similar to the one above:

$$x(y(w)) - \lambda(x)y(w) = y(x - \lambda(x))(w) + [xy](w) \in y(W_{i-1}) + W_i \subseteq W_i$$

proving the claim by induction. (It is clear when $i = 0$.) $\square$

Exercise 3.2.4. (1) Show that $\mathfrak{t}(n, F)$ is solvable.

(2) Show that any solvable subalgebra of $\mathfrak{gl}(n, \mathbb{C})$ is, up to isomorphism, isomorphic to a subalgebra of $\mathfrak{t}(n, \mathbb{C})$.

(3) If J is an ideal in L then show that L is solvable if and only if J and L/J are solvable.

(4) Prove that the following are equivalent.

(a) L is solvable.

(b) L has a sequence of ideals $L \supset J_1 \supset J_2 \cdots \supset J_n = 0$ so that J_k/J_{k+1} is abelian for each k.

(c) L has a sequence of subalgebras $L \supset L_1 \supset L_2 \cdots \supset L_n = 0$ so that L_{k+1} is an ideal in L_k and L_k/L_{k+1} is abelian for all k.