4. Jordan decomposition and Cartan's criterion

Today I will explain Cartan's criterion which implies that a Lie algebra is solvable. It uses Engel's Theorem (a Lie algebra is nilpotent iff every element is ad-nilpotent) and the Jordan decomposition. The proof requires the ground field F to be the complex numbers.

4.1. **Jordan decomposition.** For this we need the ground field F to be algebraically closed. Suppose that $x \in \mathfrak{gl}(V)$ where V is a finite dimensional vector space over F. Then V has a basis with respect to which x is in *Jordan canonical form*. Here is an example to remind you how it looks.

$$x = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} J_3(\lambda_1) & 0 \\ 0 & J_2(\lambda_2) \end{bmatrix} = J_3(\lambda_1) \oplus J_2(\lambda_2)$$

The *Jordan decomposition* of x is given by writing x as a sum of two matrices: $x = x_s + x_n$ where x_s is "semisimple" (explained below) and x_n is nilpotent:

$$x_s = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}, \quad x_n = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Furthermore, x_s and x_n commute. This is obvious by looking at the matrices. What is not obvious is that this decomposition is unique.

4.1.1. *generalized eigenspaces.* In this example there are two *Jordan blocks* $J_3(\lambda_1), J_2(\lambda_2)$ where $J_m(\lambda)$ is an $m \times m$ matrix with λ along the diagonal, 1 on the super-diagonal and 0 elsewhere. Note that the minimal polynomial of x in this example is

$$p(T) = (T - \lambda_1)^3 (T - \lambda_2)^2.$$

This corresponds to a decomposition of V as a direct sum $V = V_1 \oplus V_2$ where V_i are *generalized eigenspaces* of x defined as follows.

$$V_i = \{v \in V \mid (x - \lambda_i)^m = 0 \text{ for some } m\}.$$

The exponents $3, 2$ correspond to the size of the Jordan blocks in this case. In general we have the following proposition which follows from the fact that $F[T]$ is a principal ideal domain.

Proposition 4.1.1. *Let $x \in \mathfrak{gl}(V)$ and let $p(T) = \prod (T - \lambda_i)^{m_i}$ be the minimal polynomial of x. Then $V = \bigoplus V_i$ where V_i is a the generalized eigenspace of x corresponding to the eigenvalue λ_i. Furthermore we have the following for each i:*

(1) $x(V_i) \subseteq V_i$.
(2) *V_i is the kernel of $(x - \lambda_i)^{m_i} : V \to V$.* □

4.1.2. *semisimple endomorphisms.*

Definition 4.1.2. $x \in \mathfrak{gl}(V)$ is *semisimple* if any of the following equivalent conditions is satisfied.

(1) The roots of the minimal polynomial of x are distinct. (In other words, the exponents m_i are equal to 1.)
(2) Each generalized eigenspace is equal to the actual eigenspace $V_{\lambda_i} = \{v \in V \mid x(v) = \lambda_i v\}$.
(3) x is *diagonalizable*, i.e., there is a basis for V (consisting of eigenvectors) with respect to which the matrix of x is diagonal.

Lemma 4.1.3. *If x, y are two commuting semisimple endomorphisms of V then $x + y$ is also semisimple.*

Proof. Let V_i be the λ_i-eigenspace of x.

Claim $y(V_i) \subseteq V_i$.

Pf: For any $v \in V_i$ we have: $x(y(v)) = y(x(v)) = y(\lambda_i v) = \lambda_i y(v)$. So, $y(v) \in V_i$.

The minimal polynomial of $y|V_i$ divides the minimal polynomial of y on V. Therefore, $y|V_i$ is semisimple and thus diagonal wrt some basis for each V_i. But $x|V_i$, being multiplication by a scalar λ_i, is diagonal wrt any basis on V_i. So, $x + y$ is diagonal with respect to this basis. So, $x + y$ is semisimple. □

Theorem 4.1.4. *If F is algebraically closed and V is finite dimensional then any $x \in \mathfrak{gl}(V)$ can be written uniquely as a sum $x = x_s + x_n$ where x_s is semisimple, x_n is nilpotent and x_s, x_n commute.*

We have the following basis-independent description of x_s, the semisimple part of x as given by the Jordan canonical form. x_s is the unique endomorphism of V which sends each V_i to V_i by multiplication by λ_i and $x_n = x - x_s$.

Lemma 4.1.5. *Let $V_1, \cdots, V_k$ be the generalized eigenspaces of x and let $\mu_i \in F$. Then there is a polynomial $q(T) \in F[T]$ so that $q(x)$ sends each V_i to V_i by multiplication by μ_i.*

Proof that Lemma implies Theorem. Let $\mu_i = \lambda_i$. Then we see that $x_s = q(x)$ is a polynomial in x. This implies that x_s commutes with x and therefore with $x_n = x - x_s$.

Suppose that $x = x_s' + x_n'$ is another decomposition of x into a semisimple part x_s' and nilpotent part x_n' so that x_s', x_n' commute. Then x_s' commutes with x and therefore with $x_s = q(x)$. Therefore, by the previous lemma, $x_s' - x_s$ is semisimple. But

$$x_s' - x_s = x_n - x_n'$$

is also nilpotent since x_n, x_n' are commuting nilpotent endomorphisms of V. But the only nilpotent diagonal matrix is the zero matrix. So, $x_s = x_s'$ and $x_n = x_n'$. □

Proof of Lemma. We use the Chinese remainder theorem which says that, since $(T - \lambda_i)^{m_i}$ are relatively prime, the projection map

$$F[T] \to \prod F[T]/((T - \lambda_i)^{m_i})$$

is surjective. So, there exists a $q(T) \in F[T]$ so that $q(T) - \mu_i$ is divisible by $(T - \lambda_i)^{m_i}$ for each i. But then $q(x) - \mu_i$ is equal to 0 on V_i. □

4.2. **Cartan's criterion.** This requires $F = \mathbb{C}$. The idea is to find some condition which implies that every element of $[LL]$ is ad-nilpotent. By Engel's Theorem this will imply that $[LL]$ is a nilpotent algebra and thus a solvable algebra. And this will imply that L is solvable.

Lemma 4.2.1. *Suppose that $A \subseteq B \subseteq \mathfrak{gl}(V)$, $V \cong \mathbb{C}^n$, and*

$$M = \{x \in \mathfrak{gl}(V) \,|\, [x, B] \subseteq A\}$$

Suppose that $x \in M$ so that $\mathrm{Tr}(xy) = 0$ *for all $y \in M$. Then x is nilpotent.*

Proof. We want to show that all the eigenvalues λ_i of x are equal to zero. Let V_i be the generalized eigenspace of x corresponding to λ_i. Suppose V_i has dimension d_i. By Lemma 4.1.5 there exists $q(T) \in \mathbb{C}[T]$ so that $y = q(x)$ sends V_i to V_i by multiplication by the complex conjugate $\overline{\lambda}_i$ of λ_i for each i. Since $y = q(x)$ is a polynomial in x, $y \in M$. Therefore, $\mathrm{Tr}(xy) = 0$ by assumption. But V_i is the generalized eigenspace of xy with nonnegative real eigenvalue $\lambda_i\overline{\lambda}_i$. So,

$$\mathrm{Tr}(xy) = \sum d_i \lambda_i \overline{\lambda}_i = 0$$

implies that $\lambda_i = 0$ for all i. Therefore x is nilpotent. □

Lemma 4.2.2. *If $x, y, z \in \mathfrak{gl}(V)$ then* $\mathrm{Tr}(x[yz]) = \mathrm{Tr}([xy]z)$.

Proof. $x[yz] - [xy]z = xyz - xzy - xyz + yxz = yxz - xzy = [y, xz]$ has trace zero. □

Theorem 4.2.3 (Cartan's criterion)**.** *Suppose that $L \subseteq \mathfrak{gl}(V)$ where $V \cong \mathbb{C}^m$. Suppose* $\mathrm{Tr}(xy) = 0$ *for all $x \in [LL], y \in L$. Then L is solvable.*

Proof. We want to show that every $x \in [LL]$ is nilpotent using the lemma. So, let $A = [LL], B = L$ in the lemma. Then

$$M = \{z \in \mathfrak{gl}(V) \,|\, [zL] \subseteq [LL]\}.$$

To apply the lemma we need to show that $\mathrm{Tr}(xz) = 0$ for all $z \in M$. But $x \in [LL]$ is a linear combination of commutators $[ab]$ and $\mathrm{Tr}([ab]z) = \mathrm{Tr}(a[bz]) = 0$ since $a \in L$ and $[bz] \in [LL]$. Therefore, $\mathrm{Tr}(xz) = 0$ for all $z \in M, x \in [LL]$. The Lemma implies that every element of $[LL]$ is nilpotent. So, $[LL]$ is nilpotent and L is solvable. □

Corollary 4.2.4. *Suppose that L is a finite dimensional complex Lie algebra with the property* $\mathrm{Tr}(\mathrm{ad}\,x\,\mathrm{ad}\,y) = 0$ *for all $x \in [LL], y \in L$. Then L is solvable.*

Proof. By Cartan's criterion, the image of the adjoint representation $\mathrm{ad} : L \to \mathfrak{gl}(L)$ is solvable. Since the kernel $Z(L)$ is abelian, we conclude that L is solvable. □

Exercise 4.2.5.
(1) Prove that $\mathrm{ad}\,x_s = (\mathrm{ad}\,x)_s$ and $\mathrm{ad}\,x_n = (\mathrm{ad}\,x)_n$.
(2) Use Cartan's criterion to show that any subalgebra of $\mathfrak{t}(n, \mathbb{C})$ is solvable.
(3) Modify the proof of Cartan's criterion so that it works over any algebraically closed field of characteristic zero or $p > \dim V$.