5. Semisimple Lie algebras and the Killing form

This section follows Procesi's book on Lie Groups. We will define semisimple Lie algebras and the Killing form and prove the following.

Theorem 5.0.9. *The following are equivalent for L a finite dimensional Lie algebra over any subfield $F \subseteq \mathbb{C}$.*

1. *L is semisimple.*
2. *L has no nonzero abelian ideals.*
3. *The Killing form of L is nondegenerate.*
4. *L is a direct sum of simple ideals.*

5.1. **Definition.** First we observe that the sum of two solvable ideals I, J in L is solvable. This follows from the fact that I and $(I+J)/I = J/(I \cap J)$ are both solvable.

Definition 5.1.1. The *solvable radical* $\operatorname{Rad} L$ of L is defined to be the sum of all solvable ideals. A Lie algebra is *semisimple* if its solvable radical is zero, i.e., if it has no nonzero solvable ideal.

Proposition 5.1.2. *L is semisimple iff L has no nonzero abelian ideals.*

Proof. If L is semisimple then it has no abelian ideals. Conversely, if L is not semisimple, then L has a solvable ideal J. Then $DJ = [JJ]$ is also an ideal in L since

$$[x[JJ]] \subseteq [[xJ]J] + [J[xJ]] \subseteq [JJ]$$

We have $D^k J = 0$. So $D^{k-1}J$ is a nonzero abelian ideal in L. □

5.2. **Killing form.** The *Killing form* $\kappa : L \times L \to F$ is defined by

$$\kappa(x, y) = \operatorname{Tr}(\operatorname{ad} x \operatorname{ad} y)$$

The Killing form is clearly symmetric: $\kappa(x, y) = \kappa(y, x)$. The Killing form is also "associative":

$$\kappa([xy], z) = \kappa(x, [yz])$$

Proof. Since $\operatorname{ad}[xy] = [\operatorname{ad} x, \operatorname{ad} y]$, we have:

$$\kappa([xy], z) = \operatorname{Tr}(\operatorname{ad}[xy] \operatorname{ad} z) = \operatorname{Tr}([\operatorname{ad} x, \operatorname{ad} y] \operatorname{ad} z) = \operatorname{Tr}(\operatorname{ad} x [\operatorname{ad} y, \operatorname{ad} z]) = \kappa(x, [yz])$$

Proposition 5.2.1. *The Killing form is invariant under any automorphism ρ of L.*

Proof. The equation $\rho[xy] = [\rho(x)\rho(y)]$ for $z = \rho(y)$ is $\rho[x, \rho^{-1}(z)] = [\rho(x)z]$ which can be rewritten as: $\operatorname{ad} \rho(x) = \rho \circ \operatorname{ad} x \circ \rho^{-1}$. So,

$$\kappa(\rho(x), \rho(y)) = \operatorname{Tr}(\operatorname{ad} \rho(x) \operatorname{ad} \rho(y)) = \operatorname{Tr}(\rho \circ \operatorname{ad} x \operatorname{ad} y \circ \rho^{-1}) = \operatorname{Tr}(\operatorname{ad} x \operatorname{ad} y) = \kappa(x, y).$$ □

Lemma 5.2.2. *The kernel (also called the radical) of the Killing form of L is an ideal.*

The definition of the kernel of κ is:

$$S = \{x \in L \mid \kappa(x, z) = 0 \text{ for all } z \in L\}$$

Proof. Suppose $x \in S$ and $y \in L$. Then $\kappa([xy], z) = \kappa(x, [yz]) = 0$. □

Lemma 5.2.3. *Every abelian ideal in L is contained in the kernel of its Killing form.*

Proof. Suppose $J \subseteq L$ is an abelian ideal, $x \in J$ and $y \in L$. Then $\operatorname{ad} x \operatorname{ad} y$ sends L into J and $\operatorname{ad} x \operatorname{ad} y(J) \subseteq \operatorname{ad} x(J) = 0$. So, $(\operatorname{ad} x \operatorname{ad} y)^2 = 0$. Since nilpotent endomorphisms have zero trace, $\kappa(x, y) = \operatorname{Tr}(\operatorname{ad} x \operatorname{ad} y) = 0$ showing that $J \subseteq S$. □

Theorem 5.2.4 (Cartan)**.** *Suppose that L is a finite dimensional Lie algebra over $\mathbb{C}$. Then L is semisimple iff its Killing form is nondegenerate (its kernel $S = 0$).*

Proof. We will show that $S \neq 0$ iff L is not semisimple. If L is not semisimple then it has a nonzero abelian ideal. Any such ideal lies in the kernel S. So, $S \neq 0$.

Conversely, suppose that $S \neq 0$. Then Cartan's criterion shows that the image $\operatorname{ad}_L S$ of S under the adjoint representation $\operatorname{ad}_L : L \to \mathfrak{gl}(L)$ is solvable since

$$\kappa(x, y) = \operatorname{Tr}(\operatorname{ad} x \operatorname{ad} y) = 0$$

for all $x, y \in S$. Since $\operatorname{ad}_L S = S/Z(L)$, this implies that S is solvable. Therefore, L is not semisimple. □

Corollary 5.2.5. *Suppose that L is a finite dimensional Lie algebra over a subfield F of $\mathbb{C}$. Then L is semisimple iff its Killing form is nondegenerate.*

Proof. Suppose that the Killing form of L is nondegenerate. Then L must be semisimple since any abelian ideal is contained in the kernel of κ which is zero. Conversely, suppose that the Killing form of L has a nonzero kernel S.

Let $L_{\mathbb{C}} = L \otimes_F \mathbb{C}$ be the complexification of L. Since $L \subseteq L_{\mathbb{C}}$, it is clear that L is abelian iff $L_{\mathbb{C}}$ is abelian. This implies the L is solvable iff $L_{\mathbb{C}}$ is solvable since $D(L_{\mathbb{C}}) = (DL)_{\mathbb{C}}$. One can also see that the Killing form $\kappa_{\mathbb{C}}$ of $L_{\mathbb{C}}$ is the complexification of the Killing form κ of L. So, the kernel of $\kappa_{\mathbb{C}}$ is $S_{\mathbb{C}}$. We know that $S_{\mathbb{C}}$ is solvable from the proof of Cartan's Theorem. Therefore S is solvable and L is not semisimple. □

Example 5.2.6. Let $L = \mathfrak{sl}(2, \mathbb{R})$. This has basis

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with respect to this basis we have

$$\operatorname{ad} x = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \operatorname{ad} y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix}, \quad \operatorname{ad} h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then

$$\begin{array}{lll} \kappa(x,x) = 0 & \kappa(x,y) = 4 & \kappa(x,h) = 0 \\ & \kappa(y,y) = 0 & \kappa(y,h) = 0 \\ & & \kappa(h,h) = 8 \end{array}$$

The matrix of the form κ is therefore

$$\begin{bmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Since this matrix is invertible, κ is nondegenerate and $L = \mathfrak{sl}(2,\mathbb{R})$ is semisimple. Since this matrix has negative determinant and positive trace its *signature* (#+ eigenvalues − #− eigenvalues) is 1.

Exercise 5.2.7. Compute the Killing form of the real cross product algebra. Conclude that this algebra is semisimple but not isomorphic to $\mathfrak{sl}(2,\mathbb{R})$.

5.3. Product of simple ideals.

Theorem 5.3.1. *Suppose that L is a finite dimensional semisimple Lie algebra over any subfield $F \subseteq \mathbb{C}$. Then L can be expressed uniquely as a product of simple ideals.*

Proof. First we should point out that if L is a direct sum of two ideals $L = J_1 \oplus J_2$ then $L \cong J_1 \times J_2$ with the isomorphism given by the projection maps $L \to L/J_2, L \to L/J_1$.

If L is simple, the statement trivially holds. Otherwise, let $J \subseteq L$ be a minimal ideal. Define $J^\perp$ to be the set of all $x \in L$ so that $\kappa(x, J) = 0$. Then $J^\perp$ is an ideal since $\kappa([xy], J) = \kappa(x, [yJ]) \subseteq \kappa(x, J) = 0$ for all $x \in J^\perp, y \in L$. Therefore, $J \cap J^\perp$ is also an ideal. By minimality of J we have either $J \cap J^\perp = J$ or $J \cap J^\perp = 0$. The first case is not possible since $\kappa = 0$ on $J \cap J^\perp$ which, by Cartan's criterion, would imply that $J_\mathbb{C}$ is solvable, so J would be solvable.

Since κ is nondegenerate, we have $\dim J + \dim J^\perp = \dim L$. Therefore, $L = J \oplus J^\perp$. By induction, $J^\perp$ is a product of simple ideals. So, $L = J_1 \oplus J_2 \oplus \cdots \oplus J_n \cong \prod J_i$. To prove uniqueness of this decomposition, suppose that I is another minimal ideal. Then

$$I = [IL] = [IJ_1] \oplus [IJ_2] \oplus \cdots \oplus [IJ_n]$$

One of these summands must be nonzero. Say $[IJ_i] \subseteq I \cap J_i \neq 0$. Then $I = J_i$. □

Corollary 5.3.2. *Let L be a finite dimensional Lie algebra over $F \subseteq \mathbb{C}$. Then L is semisimple iff it is a product of simple ideals.*

Exercise 5.3.3.
(1) Show that L is nilpotent iff its Killing form is identically zero.
(2) Show that the Killing form of a nonabelian 2-dimensional Lie algebra is nontrivial.
(3) For $F \subseteq \mathbb{C}$ show that L is solvable iff $[LL]$ is contained in the kernel of κ.
(4) Show that κ nondegenerate implies L semisimple over any field.
(5) For *char* $F = 3$ show that $\mathfrak{sl}(3, F)$ modulo its center is semisimple but its Killing form is degenerate.