6. Completely reducible representations

6.1. Modules.

Definition 6.1.1. A *representation* of L is a homomorphism $L \to \mathfrak{gl}(V)$. Then V is called a *module* over L. The action of $x \in L$ on $v \in V$ is denoted $x.v$.

One can describe an action abstractly by saying that $(x, v) \mapsto v$ is F-bilinear in x, v and satisfies the property:

$$[xy].v = x.y.v - y.x.v$$

for all $x, y \in L, v \in V$.

Humphreys points out that V is a module over the associative algebra A_V generated by the image of L in $\mathfrak{gl}(V) = \mathrm{End}_F(V)$. Therefore, any submodule or quotient module of V (or any tensor power) is an L-module. Since $V^* = \mathrm{Hom}(V, F)$ is a right A_V-module, it becomes a *right L-modules*. However, L is isomorphic to its opposite L^{op} since it has an anti-automorphism given by $x \mapsto -x$. So, this explains why V^* is also a (left) L-module.

Definition 6.1.2. If V, W are L-modules then $V \times W = V \oplus W$ is the L-module with action of L given by

$$x.(v, w) = (x.v, x.w)$$

The action of L on $V \otimes W$ is defined by

$$x.(v \otimes w) = x.v \otimes w + v \otimes x.w$$

The action on $\mathrm{Hom}_F(V, W)$ is

$$(x.f)(v) = x.(f(v)) - f(x.v)$$

$$\mathrm{Hom}_L(V, W) = \{f \in \mathrm{Hom}_F(V, W) \mid x.f = 0\}$$

The action of L on the dual $V^* = \mathrm{Hom}(V, F)$ is given by

$$(x.f)(v) = -f(x.v)$$

since we take the trivial action: $x.a = 0$ of L on F.

Proposition 6.1.3. *If L is semisimple then any one-dimensional representation is trivial.*

This follows from the more general:

Lemma 6.1.4. *If L is semisimple then any homomorphism $\varphi : L \to \mathfrak{gl}(V)$ has image in $\mathfrak{sl}(V)$.*

Proof. $L = [LL]$. So, $\varphi(L) = \varphi[LL] \subseteq [\mathfrak{gl}(V)\mathfrak{gl}(V)] = \mathfrak{sl}(V)$. □

6.2. Schur's Lemma.

A module V is called *irreducible* if it has no nonzero proper submodules.

Proposition 6.2.1 (Schur's Lemma)**.** *Suppose that F is algebraically closed. Then any endomorphism of an irreducible representation V is a scalar multiple of the identity map.*

Proof. Let $f : V \to V$ be any morphism. Let λ be any eigenvalue of f. Then $f - \lambda id_V$ is an endomorphism of V with nonzero kernel. But the kernel of any morphism is a submodule of V. Since V is irreducible, it must be all of V. So, $f = \lambda id_V$. □

Definition 6.2.2. A module is *completely reducible* if it is a finite direct sum of irreducible representations.

The main theorem is Weyl's Theorem which says that any finite dimensional representation of a semisimple Lie algebra is completely reducible. This requires the Casimir operator

6.3. **Casimir operator.** Given a nongenerate symmetric bilinear form $\beta : L \times L \to F$ which is associative ($\beta([xy], z) = \beta(x, [yz])$) and any V modules we will construct an L-homomorphism $c_\beta : V \to V$. Thus c_β is an operator on every L-module. (It is a central element of the universal enveloping algebra.) The definition uses a basis for L. We need to show it is independent of the choice of basis.

Definition 6.3.1. Choose a basis $\{x_i\}$ for L and choose a dual basis $\{y_i\}$ for L with respect to the form β. Thus y_i are uniquely determined by the property:

$$\beta(x_i, y_j) = \delta_{ij}$$

Given any representation $\varphi : L \to \mathfrak{gl}(V)$, the *Casimir operator* c_β is the endomorphism of V given by

$$c_\beta(v) = \sum_i \varphi(x_i)\varphi(y_i)(v)$$

Note that $\sum_i \varphi(x_i)\varphi(y_i) \in \mathfrak{gl}(V)$ may not be an element of $\varphi(L)$.

Proposition 6.3.2. *c_β is independent of the choice of x_i.*

Proof. Any other basis is given by $x_i' = \sum a_{ij} x_j$ where $(a_{ij}) \in GL(n, F)$. The dual basis is given by $y_j' = \sum b_{jk} y_k$ where $(b_{jk}) = (a_{ij})^{-t}$ (inverse transpose). This gives the same operator c_β. □

To prove the key property (next proposition) we need the following observation. We can express any $z \in L$ in terms of both bases: $z = \sum a_i x_i = \sum b_j y_j$ and the formula for a_i, b_j is

$$a_i = \beta(z, y_i), \quad b_i = \beta(x_i, z)$$

Thus

$$z = \sum \beta(z, y_i) x_i = \beta(x_i, z) y_i$$

Proposition 6.3.3. *$c_\beta : V \to V$ is a homomorphism of L-modules. (I.e., $[z.c_\beta] = 0$ for all $z \in L$.)*

Proof. $z.c_\beta = \varphi(z) \sum \varphi(x_i)\varphi(y_i) - \sum \varphi(x_i)\varphi(y_i)\varphi(z) =$

$$\sum \varphi[zx_i]\varphi(y_i) + \sum \varphi(x_i)\varphi[zy_i]$$

$$= \sum \beta([zx_i], y_i)\varphi(x_i)\varphi(y_i) + \beta(x_i, [zy_i])\varphi(x_i)\varphi(y_i)$$

This $= 0$ since each summand is zero by associativity of β:

$$\beta([zx_i], y_i) = -\beta([x_i z], y_i) = -\beta(x_i, [zy_i])$$

□

6.4. **Weyl's Theorem.**

Theorem 6.4.1 (Weyl)**.** *Any finite dimension representation $\varphi : L \to \mathfrak{gl}(V)$ of a semisimple Lie algebra L is completely reducible.*

Lemma 6.4.2. *A representation V is completely reducible iff every submodule W has a complementary module X so that $V = W \oplus X$.* □

The idea of the proof is to use the Casimir operator c_β corresponding to the bilinear form

$$\beta(x, y) = \mathrm{Tr}(\varphi(x)\varphi(y))$$

Since L is semisimple, this is nondegenerate (and symmetric and associative). Then

$$\mathrm{Tr}(c_\beta) = \mathrm{Tr}(\sum \varphi(x_i)\varphi(y_i)) = \sum \beta(x_i, y_i) = \dim L$$

By induction on $\dim L$ we may assume that φ is a faithful representation.

Proof of Weyl's Theorem. Given any submodule W of V we will show that there is a complementary submodule Z.

Claim 1 We may assume that W is irreducible. (I.e., if W is not irreducible then it has a complement.)

Pf: We know by induction that W is completely reducible: $W = \sum W_i$ where W_i are irreducible. If there is more than one component then we get a nontrivial decomposition $W = X \oplus Y$. Then $W/X \subseteq V/X$. So, by induction there is a complementary module C/X. Then $W + C = V$ and $W \cap C = X$ since C is smaller than V, the submodule $X \subseteq C$ has a complement Z making

$$V = Y \oplus X \oplus Z = W \oplus Z$$

Proving that Z is a complement for W. Therefore, we may assume W is irreducible.

Case 1 Suppose that V/W is 1-dimensional. Then we proved that L acts trivially on V/W. This implies that the Casimir operator c_β also acts trivially on V/W. In other words $c_\beta(V) \subseteq W$. Since W is irreducible, c_β is multiplication by a scalar on W. Since $\mathrm{Tr}(c_\beta) = \dim L \neq 0$, this scalar is nonzero. Then we get $V = W \oplus \ker c_\beta$ as desired.

Case 2 In the general case, we consider

$$\mathcal{V} = \{f \in \mathrm{Hom}_F(V, W) \mid f|W \text{ is multiplication by a scalar}\}$$
$$\mathcal{W} = \{f \in \mathrm{Hom}_F(V, W) \mid f|W = 0\} \subseteq \mathcal{V}$$

The following calculation shows that $x.f \in \mathcal{W}$ for all $f \in cV$. So$\mathcal{V}$ is an L-submodule of $\mathrm{Hom}_F(V, W)$ and $\mathcal{W}$ is a codimension 1 submodule.

$$x.f(w) = x(f(w)) - f(x.w) = \lambda x.w - \lambda x.w = 0$$

Therefore, by Case 1, there is a complementary one dimensional submodule for $\mathcal{W}$ in $\mathcal{V}$. It is generated by one element $f \in \mathcal{V}$. Since 1 dim reps of L are trivial, $x.f = 0$ for all $x \in L$. This means $f \in \mathrm{Hom}_L(V, W)$. So, f is a multiple of a retraction of V to W and $\ker f$ is a complement for W in V. □