6.5. **Preservation of Jordan decomposition.** *From now on, we will assume that F is algebraically closed of characteristic zero.*

The following theorem is crucial to the next section.

Theorem 6.5.1. *Suppose $L \subseteq \mathfrak{gl}(V)$ is semisimple and V is finite dimensional. Then L contains the semisimple and nilpotent parts of its Jordan decomposition: $x_s, x_n \in L$ for all $x \in L$.*

Remark 6.5.2. The decomposition $x = x_s + x_n$ is unique and therefore does not depend on the representation. The proof is that the endomorphism $\mathrm{ad}_L x$ of L decomposes uniquely as $\mathrm{ad}_L x = (\mathrm{ad}_L x)_s + (\mathrm{ad}_L x)_n = \mathrm{ad}_L x_s + \mathrm{ad}_L x_n$. So, $\mathrm{ad}_L x_s$ is the semisimple part of $\mathrm{ad}_L x$. But the adjoint representation of L is faithful since L is semisimple. Therefore, $\mathrm{ad}_L x_s$ determines x_s uniquely.

Proof. We know that $x_s, x_n \in \mathfrak{gl}(V)$ are polynomials in x. Thus, x_s, x_n leave invariant any L-submodule W of V. Furthermore, we claim:

$$x_s|W \in \mathfrak{sl}(W)$$

This follows from the calculation:

$$\mathrm{Tr}(x_s|W) = \mathrm{Tr}(x|W) - \mathrm{Tr}(x_n|W) = 0$$

since x_n is nilpotent and $L \to \mathfrak{gl}(W)$ has image in $\mathfrak{sl}(W)$ since L is semisimple.

Also, $\mathrm{ad}\, x_s = (\mathrm{ad}\, x)_s$ and $\mathrm{ad}\, x_n = (\mathrm{ad}\, x)_n$ are polynomials in $\mathrm{ad}\, x$. So, $[x_s, L] \subseteq L$ and $[x_n, L] \subseteq L$.

Let L^* be the set of all $y \in \mathfrak{sl}(V)$ so that $[y, L] \subseteq L$ and $y|W \in \mathfrak{sl}(W)$ for all L-submodules W of V. We have that $L \subseteq L^*$ and $x_s, x_n \in L^*$ for all $x \in L$. So, it suffices to show that $L^* = L$.

But, L^* is a module over L and the condition $[y, L] \subseteq L$ implies that the action of L sends L^* into L. By Weyl's Theorem, L has a complement M in L^* and the action of L on M is trivial. Let $y \in M$. Then $[y, L] = 0$. This means that $y : V \to V$ is a homomorphism of L-modules. By definition, y preserves each submodule W of V. Take a decomposition of V into irreducible submodules W. By Schur's Lemma, y acts by multiplication by a scalar λ on each such W. But $\mathrm{Tr}(y|W) = \lambda \dim W = 0$ implies that $\lambda = 0$. Since this holds on each component of V, $y = 0$. So, $M = 0$ and $L^* = L$. So, $x_s, x_n \in L^* = L$. □

7. Representations of $\mathfrak{sl}(2, F)$

Let $L = \mathfrak{sl}(2, F)$. Recall that the standard basis is:

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and $[hx] = 2x, [hy] = -2y, [xy] = h$.

Since $\mathrm{ad}_L h$ is semisimple, it follows from Theorem 6.5.1 and the Remark after the theorem that $\varphi(h)$ is semisimple for any finite dimensional representation $\varphi : L \to \mathfrak{gl}(V)$. Therefore, V decomposes into a direct sum of eigenspaces

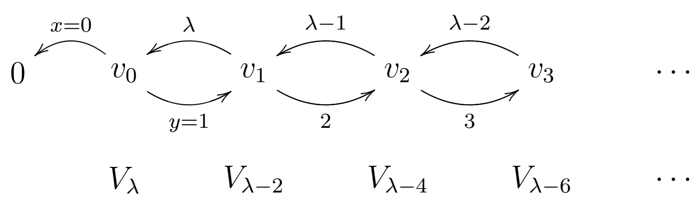
$$V_\lambda = \{v \in V \mid h.v = \lambda v\}$$

of the action of h on V. If $V_\lambda \neq 0$ then V_λ is called a *weight space* and the decomposition $V = \oplus V_\lambda$ is called the *weight space decomposition* of V.

Lemma 7.0.3. *If $v \in V_\lambda$ then $x.v \in V_{\lambda+2}$ and $y.v \in V_{\lambda-2}$.*

If V is finite dimensional, it follows that $x^n.v = 0$ for some positive n. We say that v_0 is a *maximal vector* if $v_0 \in V_\lambda$ for some λ and $x.v_0 = 0$. If $x^n.v \neq 0$ and $x^{n+1}.v = 0$ then $x^n.v$ is a maximal vector.

Lemma 7.0.4. *Suppose that $v_0 \in V_\lambda$ is a maximal vector for the action of L on V. For all $i \geq 0$ let $v_i = \frac{1}{i!}y^i.v_0$ and let $v_{-1} = 0$. Then, for all $i \geq 0$ we have:*
(1) $h.v_i = (\lambda - 2i)v_i$ *i.e.,* $v_i \in V_{\lambda-2i}$
(2) $y.v_i = (i+1)v_{i+1}$
(3) $x.v_i = (\lambda - i + 1)v_{i-1}$



Proof. (1) follows from the previous lemma. (2) follows directly from the definition of v_i. (3) is an easy induction. If $i = 0$ it is true since $x.v_0 = 0$ by assumption. Assuming (3) holds for i we get

$$(i+1)x.v_{i+1} = x.y.v_i = (h+yx).v_i = (\lambda-2i)v_i + y.(\lambda-i+1)v_{i-1} = (\lambda-2i+i(\lambda-i+1))v_i$$

But

$$\lambda - 2i + i(\lambda - i + 1) = i\lambda - i^2 + \lambda - i = (i+1)(\lambda - i)$$

Therefore,

$$x.v_{i+1} = (\lambda - i)v_i$$

showing that (3) holds for all $i \geq 0$. □

Since V is finite dimensional, and the v_i are linearly independent by (1), we have that $v_i = 0$ for sufficiently large i. Let m be maximal so that $v_m \neq 0$. Then $v_{m+1} = 0$. So:

$$x.v_{m+1} = 0 = (\lambda - m)v_m$$

This implies that $\lambda = m$.

Example 7.0.5. Take $V = F^2$ with the standard action of $L - \mathfrak{sl}(2, F)$. The basis vectors are $e_1 = \binom{1}{0} \in V_1$ and $e_2 = \binom{0}{1} \in V_{-1}$. Since $x.e_1 = 0$ we have $v_0 = e_1$. Furthermore, $y.e_1 = e_2$. Therefore, $v_1 = e_2$. Since $y.v_1 = 0$ we have $m = 1$.

Now consider the m-fold tensor product

$$W = V^{\otimes m} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{m}$$

Take the element $e_1 \otimes e_1 \otimes \cdots \otimes e_1$. Then

$$h(e_1 \otimes \cdots \otimes e_1) = h.e_1 \otimes e_1 \otimes \cdots + e_1 \otimes h.e_1 \otimes \cdots + \cdots = m(e_1 \otimes \cdots \otimes e_1)$$

Therefore, $e_1 \otimes \cdots \otimes e_1 \in W_m$. Since $x.(e_1 \otimes \cdots \otimes e_1) = 0$, this is a maximal vector.

In general, the lemma implies that $v_0, \cdots, v_m$ span an L-submodule of V. If V is irreducible then it must be all of V. Conversely, any irreducible L-module must be of this kind. Reexamination of the lemma shows that V is uniquely determined by the number $\lambda = m$ which is a nonnegative integer. Furthermore, every nonnegative integer m occurs by the example above.

Theorem 7.0.6. *Up to isomorphism the irreducible finite dimensional modules over* $\mathfrak{sl}(2, F)$ *are* $V(m)$ *for* $m \geq 0$ *where* $V(m)$ *is an* $m+1$ *dimensional representation generated by a maximal vector of weight* m *and*

$$V(m) = V_m \oplus V_{m-2} \oplus V_{m-4} \oplus \cdots \oplus V_{-m}$$

Corollary 7.0.7. *For any representation* V *of* $\mathfrak{sl}(2, F)$*, the weights are all integers, and* V *is uniquely determined up to isomorphism by the dimensions of the weight space* V_i*. Furthermore,* $\dim V_i = \dim V_{-i}$ *and* $\dim V_0 + \dim V_1$ *is equal to the number of irreducible components of* V*.*

Example 7.0.8. The weight space decomposition of L under the adjoint action is

$$L = L_{-2} \oplus L_0 \oplus L_2$$

This confirms what we already know: L is a simple module over itself. This weight space decomposition is called the *root space decomposition* of L.

Exercise 7.0.9. Given $\dim V_i$ for all i, find the decomposition of V as a direct sum of the irreducible modules $V(m)$.

Exercise 7.0.10. This is Exercise 7 on p.34. Let $\lambda \in F$ be any scalar and let $Z(\lambda)$ be the infinite dimensional vector space with basis $v_0, v_1, v_2, v_3, \cdots$.

(1) Show that the formulas in Lemma 7.0.4 define an action of $L = \mathfrak{sl}(2, F)$ on $Z(\lambda)$
(2) Show that every L-submodule of $Z(\lambda)$ contains a maximal vector.
(3) Show that $Z(\lambda)$ is irreducible if $\lambda + 1$ is not a nonnegative integer.
(4) If λ is a nonnegative integer then show that $V(\lambda)$ is a quotient module of $Z(\lambda)$ with kernel isomorphic to $Z(-\lambda - 2)$.

Exercise 7.0.11. Find the Casimir element for the adjoint representation of $L = \mathfrak{sl}(2, F)$.

Exercise 7.0.12. A Lie algebra L is called *reductive* if $Rad\, L = Z(L)$. For example, $L = \mathfrak{gl}(n, F)$. Show that L is reductive if and only if L is completely reducible as an L-module under the adjoint representation.