7.0.1. *avoiding computations.* In order to show that the formulas in Lemma 7.0.4 define an action of $L = \mathfrak{sl}(2,F)$ on $Z(\lambda)$, we need to show that three equations hold.

Lemma 7.0.13. *Suppose that $f(x), g(x) \in F[x]$ are polynomials of degree $\leq n$ with coefficients in a field F so that $f(a) = g(a)$ for at least $n+1$ values of a. Then $f(x) = g(x)$.*

Proof. If $f(x) \neq g(x)$ then $f(x) - g(x)$ can have at most n roots, i.e. there are at most n elements of F at which $f = g$. So, $f(a) = g(a)$ for $n+1$ values of a implies that $f(x) = g(x)$ as polynomials in x. $\square$

To prove that $Z(\lambda)$ is a module over L we need to verify three polynomial identities for each i. But we know that these identities hold for all integers λ greater than i. Therefore, by the Lemma, the identities hold for all $\lambda \in F$. So, $Z(\lambda)$ is a module as claimed.

Here is an example of an identity. We need to verify that $[xy].v_i = h.v_i$. Both sides are defined by the equations which give the action of x, y, h as multiplication by linear polynomials in λ:

$$[xy].v_i = x.y.v_i - y.x.v_i = x.(i+1)v_{i+1} - y.(\lambda - i + 1)v_{i-1}$$
$$= (\lambda - i)(i+1)v_i - i(\lambda - i + 1)v_i = ((\lambda - i)(i+1) - i(\lambda - i + 1))v_i$$
$$h.v_i = (\lambda - 2i)v_i$$

Therefore $[xy].v_i = h.v_i$ iff $(\lambda - i)(i+1) - i(\lambda - i + 1) = \lambda - 2i$ for all integers $i \geq 0$ and for all $\lambda \in F$. This is an easy calculation but the point is that we already know it is true for all $\lambda \in F$ by the lemma since it is true for infinitely many integers λ (by the existence of the modules $V(m)$ for all $m \geq 0$). There are two other similar computations to verify that $[hx].v_i = 2x.v_i$ and $[hy].v_i = -2y.v_i$. Each of these is a similar calculation which we do not need to do by the formal argument given above.