8. Root space decomposition

Now we come to root spaces and the classification of semisimple Lie algebras using Dynkin diagrams. My aim is to gloss over the combinatorics and emphasize the algebraic foundations.

First a review of the key definitions and theorems that we need.

8.0. **Review.** We assume L is a *semisimple* Lie algebra. This means $L = [LL]$ and L has no solvable ideals. So, the adjoint representation

$$L \hookrightarrow \mathfrak{gl}(L)$$

is faithful (a monomorphism).

The *Killing form* is the nondegenerate, symmetric, associative form on L given by

$$\kappa(x, y) = \mathrm{Tr}(\mathrm{ad}\, x \ \mathrm{ad}\, y)$$

For any $x \in L$ we have the *abstract Jordan decomposition* $x = x_s + x_n$ where x_s, x_n are uniquely determined by the formula $\mathrm{ad}\, x_s = (\mathrm{ad}\, x)_s$, $\mathrm{ad}\, x_n = (\mathrm{ad}\, x)_n$. For any representation $\varphi : L \to \mathfrak{gl}(V)$, $\varphi(x)_s = \varphi(x_s)$ and $\varphi(x)_n = \varphi(x_n)$. (Here is another proof of this: Take $\psi = (\mathrm{ad}, \varphi) : L \hookrightarrow \mathfrak{gl}(L \oplus V)$. Then by Thm 6.5.1, $\forall x, \exists y \in L$ so that $\psi(x)_s = \psi(y) = ((\mathrm{ad}\, x)_s, \varphi(x)_s) = (\mathrm{ad}\, y, \varphi(y))$. This implies $y = x_s$.) Since x_s, x_n are polynomials in x, anything that commutes with x also commutes with x_s and x_n. In particular $[xy] = 0$ implies $[x_s y] = [x_n y] = 0$.

Engel's Theorem If every element of a Lie algebra L is ad-nilpotent, then L is nilpotent.

Exercise 8.0.1.
1. (crucial observation) If $[xy] = 0$ then show that $\kappa(x_n, y) = 0$.
2. If C is a nilpotent Lie algebra which is not abelian then show that $Z(C) \cap [CC] \neq 0$.
3. If $S \subseteq \mathfrak{gl}(V)$ is solvable then $[SS]$ is nilpotent.

8.1. **Cartan subalgebra H.** By Engel's Theorem, the semisimple Lie algebra L has at least one element x which is not nilpotent. Then $x_s \neq 0$. The subspace spanned by x_s is an abelian subalgebra of L all of whose elements are semisimple.

Definition 8.1.1. The *Cartan subalgebra* of a semisimple Lie algebra L is defined to be a maximal abelian subalgebra of L consisting only of semisimple elements.

8.1.1. *preview.* We choose any Cartan subalgebra H of L. Before proceeding, we will list the important properties that we want to prove about H.

1. $\mathrm{ad}_L H$ is simultaneously diagonalizable. This gives us the *root space decomposition* of L:
$$L = L_0 \oplus \bigoplus_{\alpha \in H^*} L_\alpha$$
where
$$L_\alpha = \{x \in L \mid [hx] = \alpha(h)x\}$$
2. $L_0 = H = C_L(H)$, i.e., H is *self-centralizing*.

Definition 8.1.2. $\alpha \in H^*$ is called a *root* if $\alpha \neq 0$ and $L_\alpha \neq 0$. The set of roots is denoted Φ.

$\Rightarrow$ The root space decomposition is

$$\boxed{L = H \oplus \bigoplus_{\alpha\in\Phi} L_\alpha}$$

(3) Each $L_\alpha, \alpha \in \Phi$ is 1-dimensional.
(4) $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$ for all $\alpha, \beta \in H^*$.
(5) The restriction κ_H of the Killing form κ to $L_0 = H$ is nondegenerate. The dual of κ_H gives an inner product $(\cdot,\cdot)$ on H^*.
(6) H^* with inner product $(\cdot,\cdot)$ and the set of roots $\Phi \subset H^*$ completely determines the multiplication table for L.

8.1.2. *basic properties.* We prove the easy properties first.

Proposition 8.1.3. $\mathrm{ad}_L H$ *is simultaneously diagonalizable.*

Proof. Chose any basis $h_1, \cdots, h_n$ for H. Starting with h_1, choose a diagonalization of $\mathrm{ad}\, h_1$. Then $L \cong \bigoplus L_\lambda$ where L_λ is the λ eigenspace of h_1. Since H is abelian, $h_2, \cdots, h_n$ act as commuting semisimple operators on each L_λ. By induction on n, $\mathrm{ad}\, h_2, \cdots, \mathrm{ad}\, h_n$ are simultaneously diagonalizable. This gives a diagonalization of $\mathrm{ad}_L H$. $\square$

Remark 8.1.4. This proof shows that any linear combination of commuting semisimple elements of L is semisimple.

As pointed out in the preview, this gives the root space decomposition

$$L = L_0 \oplus \bigoplus_{\alpha\in H^*} L_\alpha$$

Proposition 8.1.5. (1) $[L_\alpha L_\beta] \subseteq L_{\alpha+\beta}$.
(2) *If* $\alpha \neq 0$ *then any element of* L_α *is* ad-*nilpotent.*
(3) $\kappa(L_\alpha, L_\beta) = 0$ *if* $\alpha + \beta \neq 0$.

Proof. (1) Suppose $x \in L_\alpha, y \in L_\beta$ and $h \in H$. Then

$$\mathrm{ad}\, h[xy] = [\mathrm{ad}\, h(x), y] + [x, \mathrm{ad}\, h(y)] = [\alpha x, y] + [x, \beta y] = (\alpha + \beta)[xy]$$

So, $[xy] \in L_{\alpha+\beta}$.

(2) $\mathrm{ad}\, x$ is nilpotent since it sends L_β to $L_{\beta+\alpha}$.

(3) Since $\alpha + \beta \neq 0$ there is some $h \in H$ so that $\alpha(h) + \beta(h) \neq 0$. Then for any $x \in L_\alpha, y \in L_\beta$ we have:

$$\alpha(h)\kappa(x, y) = \kappa([hx], y) = -\kappa(x, [hy]) = -\beta(h)\kappa(x, y)$$

So, $(\alpha(h) + \beta(h))\kappa(x, y) = 0$ making $\kappa(x, y) = 0$. $\square$

Corollary 8.1.6. *The restriction of* κ *to* $L_0 = C_L(H)$ *is nondegenerate.*

Proof. We know that κ is nondegenerate on L. So, $\kappa(h, L) = 0$ iff $h = 0$. If the restriction of κ to L_0 is degenerate then there is an $h \in L_0$ so that $\kappa(h, L_0) = 0$. But $\kappa(h, L_\alpha) = 0$ for all $\alpha \in \Phi$ by (3) above. So, $\kappa(h, L) = 0$ making $h = 0$. $\square$

8.2. **Centralizer of H.** Let $C = L_0 = C_L(H)$. We want to show that $C = H$. Since H is abelian, we know that $H \subseteq C$.

Theorem 8.2.1. *Any Cartan subalgebra of L is self-centralizing: $H = C_L(H)$.*

Proof. The outline in Humphreys works even though we used a different definition of H. Let $C = C_L(H)$. Then $H \subseteq Z(C)$.

Claim 1. If $x \in C$ then $x_s, x_n \in C$.

Pf: $x \in C$ iff $[xH] = 0$. This implies $[x_sH] = 0$ and $[x_nH] = 0$ as I pointed out in the review. So, $x_s, x_n \in C$.

Claim 2. All semisimple elements of C lie in H.

Pf: If $x = x_s \in C$ then the span of x, H is an abelian subalgebra of L all of whose elements are semisimple by Remark 8.1.4. By maximality of H, $x \in H$.

Claim 3. The restriction of κ to H is nondegenerate.

Pf: Suppose that $h \in H$ so that $\kappa(h, H) = 0$. To show that $h = 0$ it suffices to show that $\kappa(h, C) = 0$. So, let $x \in C$. Then $x = x_s + x_n$ where $x_s \in H$ by Claim 2. So, $\kappa(h, x_s) = 0$. But $[hx] = 0$ implies $\kappa(h, x_n) = 0$ by Exercise 8.0.1. So, $\kappa(h, C) = 0$ making $h = 0$ since κ is nondegenerate on C.

Claim 4. C is nilpotent.

Pf: Take any element $x \in C$. Then $x = x_s + x_n$. But $x_s \in H \subseteq Z(C)$ by Claim 2. So, $\text{ad}_C x = \text{ad}_C x_n$ is nilpotent. So, C is nilpotent by Engel's Theorem.

Claim 5. $H \cap [CC] = 0$.

Pf: Since $[HC] = 0$, $\kappa([HC], C) = 0 = \kappa(H, [CC])$. Since κ is nondegenerate on H by Claim 3, $H \cap [CC]$ must be 0.

Claim 6. C is abelian.

Pf: We use (1) and (2) of Exercise 8.0.1. If C is not abelian then $Z(C) \cap [CC] \neq 0$ by part (2). Let $x \neq 0 \in Z(C) \cap [CC]$. Then $[xC] = 0$. So, $\kappa(x_n, C) = 0$ by part (1). Therefore, $x_n = 0$ since $\kappa|C$ is nondegenerate. So, $x = x_s \in H$ contradicting Claim 5.

Claim 7. $C = H$.

Pf: For any $x \in C$, $[xC] = 0$ implies $\kappa(x_n, C) = 0$. So, $x_n = 0$. Thus $x = x_s \in H$. □

Corollary 8.2.2. (1) *The restriction of κ to H is nondegenerate.*

(2) *There is a linear isomorphism $H^* \cong H$ sending $\varphi \in H^*$ to $t_\varphi \in H$ so that*

$$\varphi(h) = \kappa(t_\varphi, h)$$

for all $h \in H$. □

8.3. **Embedded $\mathfrak{sl}(2, F)$.** For any $\alpha \in \Phi$ we will find a subalgebra of L isomorphic to $\mathfrak{sl}(2, F)$ with generators $x_\alpha \in L_\alpha, y_\alpha \in L_{-\alpha}$ and $h_\alpha \in L_0 = H$.

Lemma 8.3.1. *If $x \in L_\alpha, y \in L_{-\alpha}$ then $[xy] = \kappa(x, y)t_\alpha$.*

Proof. For any $h \in H$ we have

$$\kappa(h, [xy]) = \kappa([hx], y) = \alpha(h)\kappa(x, y)$$

Since $\kappa(h, t_\alpha) = \alpha(h)$, the Lemma follows. □

Take $x \neq 0 \in L_\alpha$. Then $\kappa(x, -) = 0$ on all L_β for $\beta \neq -\alpha$. Since κ is nondegenerate, there must be some $y \in L_{-a}$ so that $\kappa(x, y) \neq 0$. This gives:

Proposition 8.3.2. *If $\alpha \in \Phi$ then $-\alpha \in \Phi$ and $[L_\alpha L_{-\alpha}] = Ft_\alpha$.* □

Next, we need to find an element $h_\alpha \in [L_\alpha L_{-\alpha}]$ so that $\alpha(h) = 2$. This will imply that

$$(\forall x \in L_\alpha)\ [hx] = \alpha(h)x = 2x, \quad (\forall y \in L_{-\alpha})\ [hy] = -2y$$

We can then take the appropriate scalar multiples of x, y to get $[xy] = h$. Then x, y, h will span a subalgebra of L isomorphic to $\mathfrak{sl}(2, F)$ and we will use the notation $x_\alpha, y_\alpha, h_\alpha$ for this choice of x, y, h. By the proposition, it suffices to prove the following.

Lemma 8.3.3. $\alpha(t_\alpha) \neq 0$.

Proof. Suppose $\alpha(t_\alpha) = 0$. Then $[t_\alpha x] = 0 = [t_\alpha y]$ for all $x \in L_\alpha, y \in L_{-\alpha}$. By the proposition above, we can choose x, y so that $[xy] = t_\alpha$. Then the span S of x, y, t_α is a solvable subalgebra of $L \subseteq \mathfrak{gl}(L)$ with $[SS] = Ft_\alpha$. This implies that $\operatorname{ad} t_\alpha$ is nilpotent as we reviewed in Exercise 8.0.1. But the Jordan decomposition is unique and the only element which is both semisimple and nilpotent is 0. So $t_\alpha = 0$ which contradicts it definition. □

By the discussion preceding the Lemma, this proves the following.

Theorem 8.3.4. *For all $\alpha \in \Phi$ there are elements $x_\alpha \in L_\alpha$, $y_\alpha \in L_{-\alpha}$ and $h_\alpha \in H$ so that $x_\alpha, y_\alpha, h_\alpha$ span a subalgebra of L isomorphic to $\mathfrak{sl}(2, F)$ and they correspond to the standard generators x, y, h. Furthermore, $\alpha(h_\alpha) = 2$.* □

Although we have some freedom in choosing x_α, y_α we must have:

$$h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$$

This follows from the fact that h_α must be some multiple of t_α and

$$\alpha(h_\alpha) = \kappa(t_\alpha, h) = 2$$

Also, note that $t_{-\alpha} = -t_\alpha$. So, we must have $h_{-\alpha} = -h_\alpha$. Also, we may choose $x_{-\alpha} = y_\alpha$ and $y_{-\alpha} = x_\alpha$.

Let $S_\alpha \cong \mathfrak{sl}(2, F)$ be the span of $x_\alpha, y_\alpha, h_\alpha$. Then L is a module over S_α. Using our knowledge of all representations of $\mathfrak{sl}(2, F)$, we will be able to give a very complete description of the structure of the semisimple Lie algebra L.