8.4. **Root strings.** This subsection is based on Erdmann and Wildon "Introduction to Lie Algebras" an undergraduate textbook which is the place to look if you don't understand something. We first review what we have so far using an example.

8.4.1. *example:* $\mathfrak{sl}(3, F)$. Let $L = \mathfrak{sl}(3, F)$. This is 8 dimensional with H being the 2-dimensional subalgebra of diagonal matrices with trace zero.

$$H = \left\{ \begin{bmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{bmatrix} : h_1 + h_2 + h_3 = 0 \right\}$$

The off-diagonal entries have an obvious basis given by x_{ij}, the matrix with 1 in the ij position and 0 elsewhere:

$$x_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad x_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad x_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and x_{21}, x_{32}, x_{31}. Note that each x_{ij} is an eigenvector: For example,

$$[hx_{12}] = hx_{12} - x_{12}h = (h_1 - h_2)x_{12}$$

So,

(1) $x_{12} \in L_\alpha$ where $\alpha(h) = h_1 - h_2$,
(2) $x_{23} \in L_\beta$ where $\beta(h) = h_2 - h_3$ and
(3) $x_{13} \in L_{\alpha+\beta}$ with $(\alpha + \beta)(h) = h_1 - h_3$.
We also have
(4) $x_{21} \in L_{-\alpha}$
(5) $x_{32} \in L_{-\beta}$
(6) $x_{31} \in L_{-\alpha-\beta}$.

This gives the root space decomposition:

$$L = H \oplus L_\alpha \oplus L_\beta \oplus L_{\alpha+\beta} \oplus L_{-\alpha} \oplus L_{-\beta} \oplus L_{-\alpha-\beta}$$

What is h_α?

$$[x_{12}x_{21}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = h_\alpha$$

This matrix is h_α since $\alpha(h_\alpha) = h_1 - h_2 = 2$. $x_a = x_{12}, y_\alpha = x_{21}$ and

$$S_\alpha = span(x_{12}, x_{21}, h_\alpha)$$

Find the decomposition of L into irreducible S_α-modules.

One component is $V(2) = L_\alpha \oplus Fh_\alpha \oplus L_{-\alpha}$. To find the others we draw the root spaces in the following pattern:

$$
\begin{array}{ccccccc}
 & L_{\alpha+\beta} & \oplus & L_\beta & & & \cong V(1) \\
 & & & & & & \\
L_\alpha & \oplus & H & \oplus & L_{-\alpha} & & \cong V(2) \oplus V(0) \\
 & & & & & & \\
 & L_{-\beta} & \oplus & L_{-\alpha-\beta} & & & \cong V(1)
\end{array}
$$

Claim: $L_{\alpha+\beta} \oplus V_\beta \cong V(1)$.

Pf: Since $x_\alpha \in L_\alpha$, $\operatorname{ad} x_\alpha$ sends $L_{\alpha+\beta}$ to $L_{2\alpha+\beta} = 0$. Therefore, $x_{13} = x_{\alpha+\beta}$ is a maximal vector. But $h_\alpha(x_{13}) = (h_1 - h_3)x_{13} = x_{13}$. So, the weight is 1. So, x_{13} generates a submodule isomorphic to $V(1)$. (Recall that $V(\lambda)$ is generated by a maximal vector with maximal weight λ and that λ is a nonnegative integer.)

Similarly, $L_{-\beta} \oplus L_{-\alpha-\beta} \cong V(1)$. This leaves only a one dimensional submodule $V(0)$ contained in H.

8.4.2. *α-root string.* We now return to the general case. Recall that we have a root space decomposition:

$$L \cong H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

Definition 8.4.1. If $\alpha \in \Phi$ and $\beta \in \Phi \cup \{0\}$, the *$\alpha$-root string through β* is the S_α-module:

$$M = \bigoplus_{c \in F} L_{\beta + c\alpha}$$

(We will prove shortly that only integer values of c can occur in any root string.)

Proposition 8.4.2. *For any $\alpha \in \Phi$, L is a direct sum of α-root strings.*

From the example we know that the α-root string though 0 is not irreducible in general.

Proposition 8.4.3. *If $\alpha \in \Phi$ then L_α is 1-dimensional. Furthermore, the only multiples of α which are roots are $\pm\alpha$.*

Proof. Consider the α-root string though 0:

$$M = H \oplus \bigoplus_{c\alpha \in \Phi} L_\alpha$$

Let $K = \ker(\alpha : H \to F)$. Then $H = K \oplus Fh_\alpha$.

Claim: K is an S_α-module.

Pf: For any $k \in K$ we have $[x_\alpha k] = -\alpha(k)x_\alpha = 0$. Similarly, $[y_\alpha k] = 0$ and $[hk] = 0$ since $h, k \in H$. So, every nonzero element of K generated an S_α-module isomorphic to $V(0)$. This implies that

$$M/K \cong W = Fh_\alpha \oplus \bigoplus_{c\alpha \in \Phi} L_{c\alpha}$$

is an S_α-module. But the weight zero part of W is $W_0 = Fh_\alpha$. So, W contains only one irreducible $V(even)$. Since we know that W contains $S_\alpha \cong V(2)$, there cannot be any other $V(even)$. So, $L_{2\alpha} = 0$. In other words, twice a root cannot be a root. But then $\frac{1}{2}\alpha$ also cannot be a root since twice of it is a root. And this implies that $W_1 = L_{\alpha/2} = 0$. So, $W = V(2)$ is irreducible. This also implies that L_α is one dimensional. □

Corollary 8.4.4. *If $\alpha, \beta \in \Phi$ and β is not $\pm\alpha$ then the α-root string through β is irreducible and has the form:*

$$V(m) \cong L_{\beta+q\alpha} \oplus L_{\beta+(q-1)\alpha} \oplus \cdots \oplus L_{\beta-r\alpha}$$

and $m = q + r$. And $\beta(h_\alpha) = q - r$ is an integer. ($\beta(h_\alpha)$ are the Cartan integers.*)*

Proof. Let M be the α-root string through β. Then $M_0 = 0$ since the root string does not go through 0. (The proof of the last proposition showed that $S_\alpha = L_\alpha \oplus Fh_\alpha \oplus L_{-\alpha}$ is the only α-root string through 0.) Therefore, M is a direct sum of $V(odd)$s. So, the α-root string has only M_{odd} but a difference of 2 in h_α-weights corresponds to a difference of α in roots since $\alpha(h_\alpha) = 2$. So, the root string contains only $L_{\beta+k\alpha}$ for integer k. One of these is M_1 and thus is 1-dimensional. So, M is irreducible.

If $M \cong V(m)$ then $M_m = L_{\beta+q\alpha}$ and $M_{-m} = L_{\beta-r\alpha}$. The dimension of M is $q+r+1 = m+1$. This implies that

$$q + r = m = (\beta + q\alpha)(h_\alpha) = \beta(h_\alpha) + 2q$$

So, $\beta(h_\alpha) = r - q \in \mathbb{Z}$. □

8.5. **Inner product on H^*.** We just proved that $\beta(h_\alpha) \in \mathbb{Z}$. We will rephrase this in terms of the inner product on H^*.

Recall that $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha,t_\alpha)}$. And, by definition of t_β we have:

$$\beta(h_\alpha) = \kappa(t_\beta, h_\alpha) = \frac{2\kappa(t_\beta, t_\alpha)}{\kappa(t_\alpha, t_\alpha)} = \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}$$

where we used the definition of the inner product on H^*:

$$(\alpha, \beta) := \kappa(t_\alpha, t_\beta).$$

Proposition 8.5.1. *The set of roots Φ spans H^*.*

Proof. Suppose not. Then there is some $h \in H$ so that $\alpha(h) = 0$ for all $\alpha \in \Phi$. But then $[hx] = \alpha(h)x = 0$ for all $x \in L_\alpha$ and $[hx] = 0$ for all $x \in H$ since H is abelian. So, h commutes with all the generators of L and is therefore in $Z(L)$. But $Z(L) = 0$ since L is semisimple. □

This means we can choose a basis for H^* consisting of roots: $\alpha_1, \cdots, \alpha_n$. If $\beta \in \Phi$ then $\beta = \sum c_i\alpha_i$ where $c_i \in F$. Recall that we are assuming F is algebraically closed with characteristic 0. Thus F contains $\mathbb{Q}$, the rational numbers.

Claim 1: $c_i \in \mathbb{Q}$.

Pf: $(\beta, \alpha_j) = \sum c_i(\alpha_i, \alpha_j)$. Therefore,

$$\frac{2(\beta, \alpha_j)}{(\alpha_j, \alpha_j)} = \sum c_i \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$$

These fractions are Cartan integers. Also the matrix (α_i, α_j) is nonsingular since the form is nondegenerate. Therefore, c_i is the ratio of two determinants of integer matrices and therefore a rational number.

This implies that the roots lie in the $\mathbb{Q}$-span of $\alpha_1, \cdots, \alpha_n$. Let $E_{\mathbb{Q}}$ denote this rational vector space.

Claim 2: The inner product $(\cdot, \cdot)$ on $E_{\mathbb{Q}}$ has rational values and is positive definite.

Pf: Take any $\lambda \in H^*$. Then

$$(\lambda, \lambda) = \kappa(t_\lambda, t_\lambda) = \mathrm{Tr}(\mathrm{ad}\, t_\lambda\, \mathrm{ad}\, t_\lambda) =_{(1)} \sum_{\beta \in \Phi} \beta(t_\lambda)^2 =_{(2)} \sum_{\beta \in \Phi} (\beta, \lambda)^2$$

where (1) follows from the fact that $\mathrm{ad}\, t_\lambda$ acts on L_β by multiplication by $\beta(t_\lambda)$ and (2) follows from the observation that $\beta(t_\lambda) = \kappa(t_\beta, t_\lambda) = (\beta, \lambda)$. Thus $(\cdot, \cdot)$ is positive semi-definite. Since it is nondegenerate, it must be positive definite.

Let $E = E_{\mathbb{Q}} \otimes \mathbb{R}$.

Theorem 8.5.2. (1) *E is a real vector space with a positive definite inner product, i.e., E is Euclidean space.*
(2) *Φ spans E and $0 \notin \Phi$.*
(3) *If $\alpha \in \Phi$ then $-\alpha \in \Phi$ and not other real multiple of α is in Φ.*
(4) *If $\alpha, \beta \in \Phi$ then $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.*
(5) *If $\alpha, \beta \in \Phi$ then*

$$\beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha \in \Phi$$

Proof. We proved (1)-(4). The class figured out the proof of (5):

$$\frac{2(\beta, \alpha)}{(\alpha, \alpha)} = \beta(h_\alpha) = r - q$$

in the notation of the α-root string through β. So,

$$\beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha = \beta + (q - r)\alpha$$

and $q \geq q - r \geq -r$. So, this is one of the roots that occur in the α string through β. $\square$