9. Abstract root systems

We now attempt to reconstruct the Lie algebra based only on the information given by the set of roots Φ which is embedded in Euclidean space E.

9.1. **Definition.** Any finite subset Φ of Euclidean space E satisfying the conditions of Theorem 8.5.2 will be called a root system. To repeat:

Definition 9.1.1. A *root system* is defined to be a subset Φ of standard Euclidean space E (i.e., finite dimensional real vector space with a positive definite inner product $(\cdot,\cdot)$) satisfying the following.

(1) Φ is finite, spans E and $0 \notin \Phi$.
(2) If $\alpha \in \Phi$ then $-\alpha \in \Phi$ and not other real multiple of α is in Φ.
(3) If $\alpha, \beta \in \Phi$ then
$$\langle \beta, \alpha \rangle := \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$$
(4) If $\alpha, \beta \in \Phi$ then
$$\beta - \langle \beta, \alpha \rangle \alpha \in \Phi$$

Note that $\langle \alpha, \alpha \rangle = 2$.

Proposition 9.1.2. *Let $\sigma_\alpha : E \to E$ be the linear mapping given by*
$$\sigma_\alpha(x) = x - \langle x, \alpha \rangle \alpha$$
Then σ_α is reflection through the hyperplane perpendicular to α.

Proof. σ_α sends α to $-\alpha$ and leaves fixed every vector perpendicular to α. □

Condition (4) says that $\sigma_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$.

Definition 9.1.3. The *Weyl group* W is defined to be the subgroup of $GL(E)$ generated by the reflections σ_α for all $\alpha \in \Phi$.

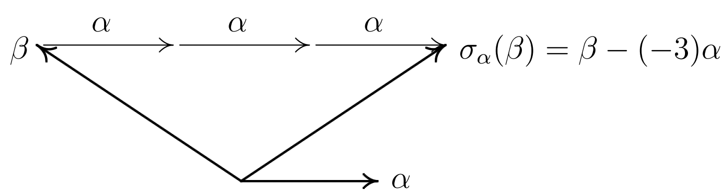


Figure 1. In this example, $\langle \beta, \alpha \rangle = -3$ and $\sigma_\alpha(\beta) = \beta + 3\alpha \in \Phi$. We will prove later that the vectors in the middle: $\beta + \alpha, \beta + 2\alpha$ are also in Φ.

Example 9.1.4. (1)
$$A_1^n = \{\pm\epsilon_i \mid 1 \le i \le n\} \quad \subseteq E = \mathbb{R}^n$$
is a root system. σ_{ϵ_i} sends ϵ_i to $-\epsilon_i$ and leaves the other unit vectors ϵ_j fixed. The matrix of this reflection map is the diagonal matrix with -1 in the ith position with other diagonal entries equal to 1. Therefore, the Weyl group is the group of diagonal matrices with ± 1 on the diagonal: $W = \mathbb{Z}_2^n = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$. ($\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$)

(2)
$$A_{n-1} = \{\pm(\epsilon_i - \epsilon_j) \,|\, 1 \le i < j \le n\}$$
is a root system in $E = \{x \in \mathbb{R}^n \,|\, \sum x_i = 0\}$. For $\alpha = \epsilon_i - \epsilon_j$, the reflection σ_α switches ϵ_i and ϵ_j and leaves the other ϵ_k fixed. This clearly sends the set A_{n-1} to itself. The reflections are transpositions. So, the Weyl group is the symmetric group S_n.

(3)
$$D_n = \{\pm(\epsilon_i \pm \epsilon_j) \,|\, 1 \le i < j \le n\}$$
is a root system. For $\alpha = \epsilon_i + \epsilon_j$, the reflection σ_α switches ϵ_i and $-\epsilon_j$ (and $-\epsilon_i \leftrightarrow \epsilon_j$) and leaves the other ϵ_k fixed. This clearly sends the set D_n to itself. One can show that the Weyl group is the group of signed permutations with an even number of negative signs.

Exercise 9.1.5. Find an explicit isomorphism $D_3 \cong A_3$.

9.2. **Two roots.**

Lemma 9.2.1. *Suppose that $\alpha, \beta \in \Phi$ and $\beta \neq \pm\alpha$. Then*
$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 0, 1, 2 \text{ or } 3.$$

Proof. This follows from the Schwartz inequality:
$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = \frac{4(\alpha,\beta)(\beta,\alpha)}{(\beta,\beta)(\alpha,\alpha)} = \frac{4(\alpha,\beta)^2}{||\beta||^2||\alpha||^2} \le 4$$
where equality holds iff α, β are collinear. □

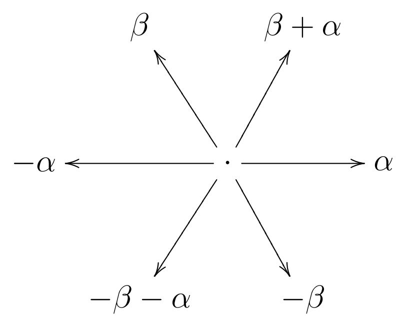
This is also equal to $4\cos^2\theta$ where θ is the angle between α, β.
$$\begin{array}{rcllll} \cos^2\theta & = & 0 & \frac14 & \frac12 & \frac34 \\ \pm\cos\theta & = & 0 & \frac12 & \frac{\sqrt2}{2} & \frac{\sqrt3}{2} \\ \theta \text{ or } \pi-\theta & = & \frac{\pi}{2} & \frac{\pi}{3} & \frac{\pi}{4} & \frac{\pi}{6} \end{array}$$

If the product $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$ is 0 then α, β are perpendicular and $\sigma_\alpha(\beta) = \beta$. This is illustrated in the first example above.

If the product of $\langle \alpha, \beta \rangle, \langle \beta, \alpha \rangle$ is nonzero, one of them must be ±1. By symmetry assume $\langle \alpha, \beta \rangle = \pm1$. Then $\langle \beta, \alpha \rangle = \pm1, \pm2, \pm3$.
$$|\langle \beta, \alpha \rangle| = \frac{\langle \beta, \alpha \rangle}{\langle \alpha, \beta \rangle} = \frac{2(\beta,\alpha)/||\alpha||^2}{2(\beta,\alpha)/||\beta||^2} = \frac{||\beta||^2}{||\alpha||^2} = 1, 2, 3$$
This looks like 6 cases, but this reduces to 3 cases with the following observation. Let
$$\gamma = \sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$$
Then
$$\langle \gamma, \alpha \rangle = \langle \beta, \alpha \rangle - 2\langle \beta, \alpha \rangle = -\langle \beta, \alpha \rangle$$
So, by replacing β with γ if necessary, we may assume that $\langle \beta, \alpha \rangle$ is *negative*.

9.2.1. *root system A_2.* Suppose that $\langle \beta, \alpha \rangle = -1$. Then $||\alpha|| = ||\beta||$ and $\theta = 2\pi/3$. Also

$$\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha = \beta + \alpha$$



This is the root system of $\mathfrak{sl}(3, F)$. Also, it is a special case of example 2 above with $n = 3$. The correspondence is

$$\alpha = \epsilon_1 - \epsilon_2, \quad \beta = \epsilon_2 - \epsilon_3 \qquad (\alpha + \beta = \epsilon_1 - \epsilon_3)$$

with $||\alpha|| = ||\beta|| = \sqrt{2}$ and

$$\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)} = \frac{-2}{2} = -1$$

In terms of H, the Cartan subalgebra of $L = \mathfrak{sl}(3, F)$, $\epsilon_i \in H^*$ is defined by $\epsilon_i(h) = h_i$. So,

$$\alpha(h) = \epsilon_1(h) - \epsilon_2(h) = h_1 - h_2$$

which agrees with the earlier terminology.

Exercise 9.2.2. Generalize this correspondenc to show that Example 2 is the root system for $\mathfrak{sl}(n, F)$.

An animation of $A_3 = D_3$, the root system of $\mathfrak{sl}(4, F)$ is on the webpage. In that rotating figure, the green arrow is $\alpha = \epsilon_1 - \epsilon_2$, the red arrow is $\beta = \epsilon_2 - \epsilon_3$ and the blue arrow is $\gamma = \epsilon_3 - \epsilon_4$. The angles are:

$$\langle \alpha, \beta \rangle = 2, \quad \theta = 2\pi/3$$

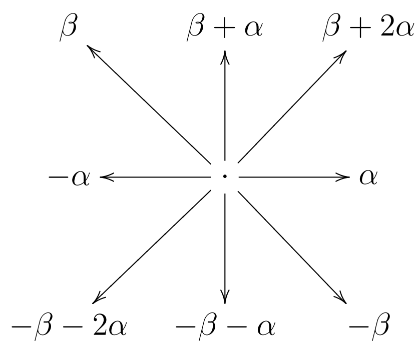
$$\langle \beta, \gamma \rangle = 2, \quad \theta = 2\pi/3$$

$$\langle \alpha, \gamma \rangle = 0, \quad \theta = \pi/2$$

The white arrow is the sum of these three roots: $\alpha + \beta + \gamma = \epsilon_1 - \epsilon_4$. Note that all roots have the same length: $\sqrt{2}$.

9.2.2. *root system* $B_2 = C_2$. Suppose that $\langle \beta, \alpha \rangle = -2$. Then $||\beta|| = \sqrt{2}||\alpha||$ and $\theta = 3\pi/4$. Also

$$\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha = \beta + 2\alpha$$



This root system has *short roots* of length 1 and *long roots* of length $\sqrt{2}$.

Exercise 9.2.3. Show that the 26 vectors in $\mathbb{R}^3$ given by

$$\pm\epsilon_i(\text{ 6 vectors}), \pm\epsilon_i \pm \epsilon_j(\text{ 12 vectors}), \pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3(\text{ 8 vectors})$$

do not form a root system. (These vectors form a $2 \times 2 \times 2$ cube just as the figure above forms a 2×2 square.)

Definition 9.2.4. The root system B_n in $\mathbb{R}^n$ is defined to be the union of the set of $2n$ short roots $\pm\epsilon_i$ and the $4\binom{n}{2}$ long roots $\pm\epsilon_i \pm \epsilon_j$.

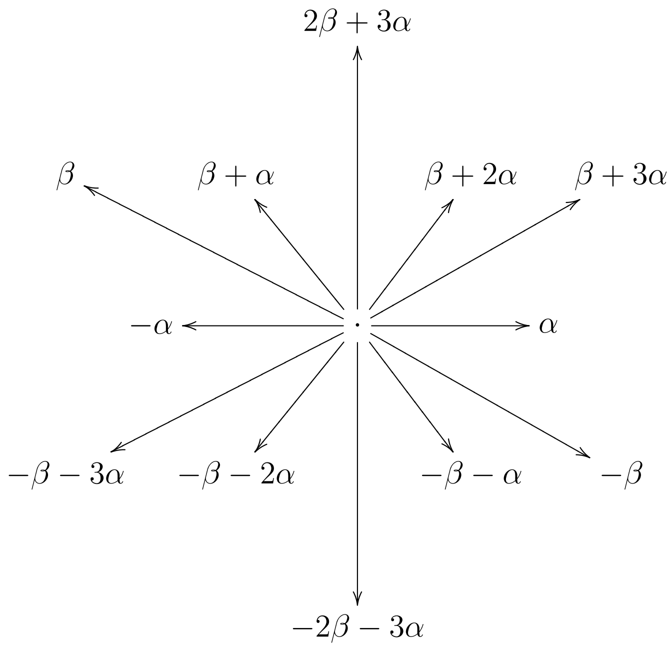
The long roots of B_n form a subsystem D_n as in Example 9.1.4 (3) and the short roots form A_1^n as in Example 9.1.4 (1). For example, for $n = 3$, there are 6 short roots and 12 long roots and these long roots form $D_3 = A_3$.

Definition 9.2.5. The root system C_n in $\mathbb{R}^n$ is defined to be the union of the set of $2n$ long roots $\pm 2\epsilon_i$ and the $4\binom{n}{2}$ short roots $\pm\epsilon_i \pm \epsilon_j$.

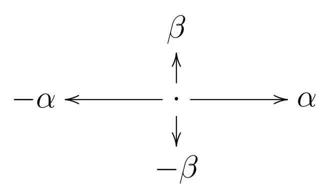
The short roots of C_n form a subsystem D_n. For $n = 3$ there are 12 short roots and 6 long roots. The case B_3, C_3 can be seen in these animations: B_3, C_3. The short roots are red and the long roots are blue in both.

9.2.3. *root system* G_2. Suppose that $\langle \beta, \alpha \rangle = -3$. Then $||\beta|| = \sqrt{3}||\alpha||$ and $\theta = 5\pi/6$. Also

$$\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha = \beta + 3\alpha$$



9.2.4. *root system* $A_1 \times A_1$. When $\langle \beta, \alpha \rangle = 0$ the two roots are perpendicular. They could have any length.



This is the root system of $\mathfrak{sl}(2, F) \times \mathfrak{sl}(2, F)$.

9.3. **Irreducible root systems.**

Definition 9.3.1. We say that a root system Φ *decomposes* if it is a disjoint union $\Phi_1 \cup \Phi_2$ of two nonempty subsets so that every root in Φ_1 is perpendicular to every root of Φ_2. We say that Φ is *irreducible* if there is no such decomposition.

If Φ decomposes, then E also decomposes as an orthogonal direct sum $E = E_1 \oplus E_2$ where E_i is the span of Φ_i. Each $\Phi_i \subset E_i$ is a root system.

Exercise 9.3.2. Show that the root system of a product $L = L_1 \times L_2$ of two semisimple Lie algebras decomposes as the union of the root systems of L_1, L_2. Conversely, any decomposition of the root system of L comes from such a factorization of L.

Proposition 9.3.3. *The decomposition of a root system Φ into irreducible components is unique.*

Proof. If $\Phi = \bigcup \Phi_i$ and $\Phi = \bigcup \Psi_j$ are any two decompositions then so is $\Phi = \bigcup \Phi_i \cap \Psi_j$. $\square$

9.4. Bases for root systems.

Definition 9.4.1. A subset $\Delta = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ of Φ is called a *base* for the root system if

(1) Δ is a basis for E
(2) Every element $\beta \in \Phi$ can be written as $\beta = \sum k_i\alpha_i$ where all k_i are integers with the same sign (or zero).

Given a choice of base Δ, the root in Δ are called *simple roots*, roots which are positive linear combinations of simple roots are called *positive roots* the set of positive roots is denoted Φ_+. The set of *negative roots* (negatives of positive roots) is Φ_-. Thus

$$\Phi = \Phi_+ \cup \Phi_-$$

Example 9.4.2. In the root systems $A_2, B_2, G_2, A_1 \times A_1$ above, the roots α, β were chosen to form a base for the root system.

Example 9.4.3. In the root system

$$A_{n-1} = \{\epsilon_i - \epsilon_j \in \mathbb{R}^n\}$$

the roots $\alpha_i := \epsilon_i - \epsilon_{i+1}$, $i = 1, \cdots, n-1$ form a base. In the A_3-animation, the simple roots are green, red and blue.

Example 9.4.4. In the second set of animations for the root systems B_3 and C_3 the simple roots have green tips, and the positive roots are in the foreground (with negative roots behind a semitransparent plane).

Lemma 9.4.5. *If $\alpha, \beta \in \Phi$ and $(\alpha, \beta) < 0$ then $\alpha + \beta$ is a root. Similarly, if $(\alpha, \beta) > 0$ then $\alpha - \beta$ is a root.*

Proof. By symmetry we may assume $\langle \alpha, \beta \rangle = \pm 1$ and this has the same sign as (α, β). Then

$$\sigma_\beta(\alpha) = \begin{cases} \alpha + \beta & \text{if } (\alpha, \beta) < 0 \\ \alpha - \beta & \text{if } (\alpha, \beta) > 0 \end{cases}$$

and $\sigma_\beta(\alpha) \in \Phi$ by definition of root system. □

Theorem 9.4.6. *Every root system has a base.*

Proof. First choose a vector x which is not perpendicular to any root. Since there are only finitely many roots and their perpendicular hyperplane have zero measure, their union also has zero measure. Take any point in the complement.

Define a root $\beta \in \Phi$ to be *positive* if $(\beta, x) > 0$. Let S be the set of all real numbers $(\beta, x) > 0$ for $\beta \in \Phi$. Then S is finite. Define a positive root to be *indecomposable* if it cannot be written as a sum of other positive roots.

Claim 1 Every positive root β can be written as a sum of indecomposable positive roots.

Pf: This holds by induction on the number of elements in the set S which are less than (β, x). If this number is zero then β is indecomposable. If β decomposes into a sum $\beta = \beta_1 + \beta_2$, we have $(\beta_i, x) < (\beta, x)$. So, by induction, each β_i is a sum of indecomposable positive roots.

Claim 2 If α, β are indecomposable positive roots then $(\alpha, \beta) \leq 0$.

Pf: If $(\alpha, \beta) > 0$ then $\gamma = \alpha - \beta$ is a root and therefore either γ or $-\gamma$ is a positive room. In the first case, $\alpha = \beta + \gamma$ is not indecomposable. In the second case, $\beta = \alpha + (-\gamma)$ is not indecomposable.

Claim 3 Let Δ be the set of indecomposable positive roots. Then Δ is a base for Φ.

Pf: Since the second condition in the definition of a base is satisfied by construction, it suffices to show that Δ is a basis for E. Since Δ spans Φ it also spans E. So, it suffices to show Δ is linearly independent.

Suppose not. Then there is a linear dependence $\sum k_i \alpha_i = 0$ where $\alpha_i \in \Delta$. Separate the positive and negative coefficients κ_i to obtain a relation:

$$\sum s_i \alpha_i = \sum t_j \alpha_j$$

where s_i, t_j are all ≥ 0 and $\alpha_i \neq \alpha_j \in \Delta$. Let z denote this sum. Then

$$(z, z) = \sum s_i t_j (\alpha_i, \alpha_j) \leq 0$$

by Claim 2. So, $z = 0$ proving Claim 3 and the theorem. □