10. Weyl group and Weyl chambers

We will use the Weyl group and the geometry of Weyl chambers to prove basic properties of root systems, such as the uniqueness of the base up to isomorphism.

Recall that the Weyl group is the subgroup W of $GL(E)$ generated by the reflections σ_β through the roots $\beta \in \Phi$. Since $\sigma_{-\beta} = \sigma_\beta$ we can restrict to positive roots β. It is important to notice that reflections and thus all elements of W are orthogonal transformations, i.e., that they preserve the inner product:

$$(w(x), w(y)) = (x, y)$$

The *Weyl chambers* are defined to be the components of the complement in E of the union of all hyperplanes perpendicular to the roots. The elements of W are orthogonal and permute the roots. Therefore, W permutes the Weyl chambers.

10.1. Simple reflections.

Choose a base Δ for Φ. Then the *simple reflections* are defined to be σ_{α_i} where α_i are the simple roots (elements of Δ).

Lemma 10.1.1. *If $\alpha \in \Delta$, the simple reflection σ_α sends α to $-\alpha$, $-\alpha$ to α and permutes all of the other positive roots.*

Proof. Suppose that β is a positive root not equal to α. Then β is not equal to a scalar multiple of α. So, in the expansion of β as a positive linear combination of simple roots: $\beta = \sum k_i\alpha_i$ where, say, $\alpha = \alpha_1$, one of the other coefficients, say $k_2 > 0$. Then

$$\sigma_\alpha(\beta) = \beta - \langle\beta, \alpha\rangle\, \alpha = (k_1 - \langle\beta, \alpha\rangle)\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n$$

is a positive root since $k_2 > 0$. □

Proposition 10.1.2. *For every $\beta \in \Phi$ there is a simple root α and a sequence of simple reflections whose composition carries α to β.*

Proof. We may assume that β is a positive root: If we know that a sequence of simple reflections carries α to β then, if we first do σ_α (sending α to $-\alpha$) and then do these same simple reflections, then we will carry α to $-\beta$.

Define the *height* of a positive root β as the sum of the coefficients k_i in the expansion $\beta = \sum k_i\alpha_i$. If the height of β is 1 then β is simple and there is nothing to prove. Then we proceed by induction.

Suppose that β is a positive root with height ≥ 2. Then at least two of the coefficients, say k_1, k_2 are positive.

<u>Claim</u> There exists a simple root α so that $(\alpha_i, \alpha) > 0$ for some α_i so that $k_i > 0$.

Pf: If not then $(\beta, \alpha) \leq 0$ for all simple roots α and therefore, $(\beta, \beta) = \sum k_i(\beta, \alpha_i) \leq 0$ which is not possible since $\beta \neq 0$.

Choose α as in the Claim. Then

$$\sigma_\alpha(\beta) = \beta - \langle\beta, \alpha\rangle\, \alpha$$

is a positive root by the lemma and it has smaller height than β since $\langle\beta, \alpha\rangle$ is positive. Therefore, by induction on height, we can do a few more simple reflections to send $\sigma_\alpha(\beta)$ to a simple root. □

Lemma 10.1.3. *If $w \in W$ and β is any root then $w\sigma_\beta w^{-1} = \sigma_{w(\beta)}$.* □

Corollary 10.1.4. *The Weyl group W is generated by simple reflections.*

Proof. For any $\beta \in \Phi$ there is a simple root α and $w \in W$, a product of simple reflections, so that $w(\alpha) = \beta$. But then the generator σ_β of W is a product of simple reflections: $\sigma_\beta = w\sigma_a w^{-1}$, proving the corollary. □

Definition 10.1.5. A *reduced expression* for w is a factorization of w as a product of k simple reflections $w = \sigma_1\sigma_2 \ldots \sigma_k$ where k is minimal. This minimal number is called the *length* of w.

Lemma 10.1.6. *For any $w \in W$, the following two numbers are equal:*

(1) *$n(w)$ = the number of positive roots β which are sent to negative roots by w.*
(2) *$\ell(w)$ = the smallest number of simple reflections whose product is w.*

Proof. Every simple reflection sends exactly on positive root to a negative root and the other positive roots to positive roots. Therefore, a product of k simple reflections can send at most k positive roots to negative roots. Therefore, $n(w) \le \ell(w)$.

To prove the lemma it suffices to show that, if w is a product of $k+1$ simple reflections $w = \sigma_0\sigma_1 \cdots \sigma_k$ and $n(w) < k+1$ then the expression is not reduced. So, suppose the expression is reduced. Then, the subword $w_1 = \sigma_1\sigma_2 \cdots \sigma_k$ is also reduced with length k. By induction on k we have $\ell(w_1) = n(w_1) = k$. So, w_1 sends k positive roots to negative roots. Since $n(w) < k+1$, the last simple reflection $\sigma_0 = \sigma_\alpha$ sends one of these negative roots back to a positive root. This root must be $-\alpha$ which is equal to some $w(\beta), \beta \in \Phi_+$. But this means the simple reflections σ_i keep β positive for a while, switches the sign then keeps it negative. So

$$w_1 = \sigma_1 \cdots \sigma_i \cdots \sigma_k = g\sigma_i h$$

where $h(\beta) = \alpha_j \in \Delta$ and $\sigma_i = \sigma_{\alpha_j}$ and $g(\alpha_j) = \alpha$ so that:

$$w_1(\beta) = g\sigma_{\alpha_j} h(\beta) = g\sigma_{\alpha_j}(\alpha_j) = g(-\alpha_j) = -\alpha$$

But this last step $g(\alpha_j) = \alpha$ implies that $g\sigma_i g^{-1} = \sigma_{g(\alpha_j)} = \sigma_\alpha$. Therefore,

$$w = \sigma_\alpha w_1 = \sigma_\alpha g \sigma_{\alpha_j} h = gh$$

showing that the original expression for w was not reduced. □

Proposition 10.1.7. *The only element of W which sends all positive roots to positive roots is the identity.* □

Corollary 10.1.8. *The only element $w \in W$ which sends Δ to Δ is $w = 1$.* □

10.2. **Weyl chambers.** We will show that there is a 1-1 correspondence between bases Δ, Weyl chambers C and elements w of the Weyl group. This is equivalent to saying that the action of W on both of these sets is simply transitive (transitive and effective).

Lemma 10.2.1. *The Weyl group acts transitively on the set of Weyl chambers.*

Proof. Whereas hyperplanes cut E up into disjoint regions, the intersection of any two hyperplanes has codimension two and the complement of any finite number of codimension 2 subspaces is connected.

If we take any two Weyl chambers C_0, C_1 then there is a path connecting C_0 to C_1 which passes through a finite number of the hyperplanes separating the chambers but does not pass through intersections of hyperplanes. If this finite number is 0 then $C_0 = C_1$ and there is nothing to prove. If this number is positive, then take the first hyperplane, say $\beta^\perp$ which crosses the path from C_0 to C_1. Then σ_β takes C_0 to the chamber C_2 on the other side of the hyperplane $\beta^\perp$. This is closer to C_1 by construction and therefore $w(C_2) = C_1$ for some $w \in W$ by induction. Then $C_1 = w\sigma_\beta(C_0)$. □

Lemma 10.2.2. *Let C be the set of all $x \in E$ with the property that $(x, \beta) > 0$ for all positive roots β. Then C is a Weyl chamber.*

We call C the *fundamental chamber.*

Proof. Clearly C is convex and therefore connected. Also C is disjoint from all hyperplanes $\beta^\perp$. Therefore, C is contained in some Weyl chamber C_0. Suppose that $y \in C_0$ then, since C_0 is connected, there is a path $\gamma(t)$ in C_0 connecting $x \in C$ to y. This path does not cross any of the hyperplanes. Therefore, by the intermediate value theorem, the sign of $(\gamma(t), \beta)$ remains unchanged. Since it starts as positive, it remains positive. So, $y \in C$ proving that $C = C_0$ is a Weyl chamber. □

The above proof has a gap: it assumes that C is nonempty. To fix this I added the following in class.

Lemma 10.2.3. *Let C' be the set of all $x \in E$ so that $(x, \alpha) > 0$ for all positive roots α. Then C' is nonempty and $C' = C$.*

Proof. It is easy to see that $C \subseteq C'$ because the condition (x, β) for all positive β implies that $(x, \alpha) > 0$ for simple α. Conversely, if $(x, \alpha) > 0$ for simple α then $(x, \beta) = \sum k_i(x, \alpha_i) > 0$ for all positive β.

To see that C' is nonempty, note that the mapping $\psi : E \to E^*$ given by $\alpha \mapsto (-, \alpha)$ is an isomorphism. Therefore it sends basis to basis. So, $\{(-, \alpha_i)\}$ is a basis for E^*. Equivalently,

$$x \mapsto [(x, \alpha_1), (x, \alpha_2), \cdots, (x, \alpha_n)] \in \mathbb{R}^n$$

gives a linear isomorphism $E \cong \mathbb{R}^n$. So, there is some $x \in E$ which goes to $[1, 1, \cdots, 1] \in \mathbb{R}^n$. Then $x \in C'$ showing that C' is nonempty. □

Theorem 10.2.4. *There is a one-to-one correspondence between bases Δ, Weyl chambers C and elements of the Weyl group W.*

Proof. We know that W acts transitively on Weyl chambers and effectively on bases. Therefore, it suffices to show that there is a 1-1 corresondence between bases and chambers which respects the action of W.

For each Δ we have the set Φ_+ of all roots which are positive linear combinations of elements of Δ. The lemma associates a corresponding fundamental chamber C. Conversely, given any chamber C, the positive roots are those with $(x, \beta) > 0$ for all $x \in C$

and the simple roots are the indecomposable positive roots. It is easy to see that this correspondence is W-equivariant, i.e., $w\Delta$ corresponds to wC. □

Exercise 10.2.5.
(1) Show that there is a unique element of the Weyl group $w_0 \in W$ of maximal length. (w_0 corresponds to $-C_0$ where C_0 is the fundamental chamber.)
(2) Suppose that $\beta = \sum k_i\alpha_i$ is a positive root with maximal height. ($\sum k_i$ is maximal).
   (a) Show that β is in the closure of C_0.
   (b) Show that any α_j with $k_j = 0$ is perpendicular to the α_i with $k_i > 0$.
(3) Suppose that Φ is an irreducible root system. Then show that there exists a unique root of maximal height. (If there are two then their difference would be a root.)