11. Classification of semisimple Lie algebras

I will explain how the Cartan matrix and Dynkin diagrams describe root systems. Then I will go through the classification using classical examples of Lie algebras following Erdmann and Wildon. I will skip the combinatorial proof. (See Humphreys. This part of the book requires no knowledge of Lie algebras!)

11.1. **Cartan matrix and Dynkin diagram.**

Definition 11.1.1. Suppose that Φ is a root system with base Δ. Let $\alpha_1, \cdots, \alpha_n$ be the list of simple roots. Then the *Cartan matrix* is defined to be the $n \times n$ matrix C with entries $\langle \alpha_i, \alpha_j \rangle$. We recall that $\langle \alpha_i, \alpha_j \rangle = \alpha_i(h_{\alpha_j})$ where h_α is the unique element of $[L_\alpha, L_{-\alpha}]$ so that $\alpha(h_\alpha) = 2$.

Example 11.1.2. If we let α_1 be the short root, the Cartan matrices for the root systems of rank 2 are:

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

$$A_1 \times A_1 \qquad A_2 \qquad B_2 = C_2 \qquad G_2$$

The diagonal entries are always equal to 2. The off diagonal entries are negative or zero.

Proposition 11.1.3. *The Cartan matrix of a semisimple Lie algebra is nonsingular. In fact, its determinant is positive and all of its diagonal minors are positive.*

Proof. By multiplying the j column of C by the positive number $(\alpha_j, \alpha_j)/2$, it becomes the matrix (α_i, α_j) which has positive determinant since it is the matrix of a positive definite symmetric form on $E \cong \mathbb{R}^n$. Any diagonal submatrix of C is proportional to the matrix of the same form on a vector subspace. This is positive definite since it is the restriction of a positive definite form. □

Definition 11.1.4. The *Dynkin diagram* of a root system of rank n is defined to be a graph with n vertices labelled with the simple roots α_i and with edges given as follows.

(1) No edge connecting roots α_i, α_j if they are orthogonal (equivalently, if $c_{ij} = 0$.)

$$\circ \qquad \circ$$

(2) A single edge connecting α_i, α_j if α_i, α_j are roots of the same length which are not orthogonal (equivalently, $c_{ij} = c_{ji} = -1$.)

$$\circ \text{---} \circ$$

(3) A double edge pointing from α_i to α_j if α_i, α_j are not perpendicular and $||\alpha_i||^2 = 2||\alpha_j||^2$

$$\circ_i \Longrightarrow \circ_j$$

(4) A triple edge pointing from α_i to α_j if α_i, α_j are not perpendicular and $||\alpha_i||^2 = 3||\alpha_j||^2$

$$\circ_i \Rrightarrow \circ_j$$

The *Coxeter graph* is the same graph without the orientation on the multiple edges.

Corollary 11.1.5. *The Coxeter graph of a root system contains no cycle of single edges.*

Proof. If the diagram contains such a cycle then the vector v which is sum of the simple roots corresponding to the vertices of the cycle have the property that $v^tCv = 0$ since the only positive number in each column is a 2 and the each simple edge incident to α_i puts a -1 in the ith column. But v^tCv is proportional to (v, v) since all simple roots in the cycle have the same length. This contradicts the fact that the inner product is positive definite. ☐

In a similar way, we can list more "forbidden subgraphs" of the Coxeter graph. The classification is given by listing all possible graphs which contain no forbidden subgraphs. This gives a list of all possible Dynkin diagrams. We need to construct semisimple Lie algebras for each diagram and show that the diagram determines the Lie algebra uniquely.

11.2. **Orthogonal algebra $\mathfrak{so}(2n, F)$.** Recall that any bilinear form $f : V \times V \to F$ defines a subalgebra of $\mathfrak{gl}(V)$ consisting of all x so that

$$f(x(v), w) + f(v, x(w)) = 0$$

We will take $V = F^n$ and take only $x \in \mathfrak{sl}(n, F)$ satisfying the above. When $V = F^n$ there is a unique $n \times n$ matrix S so that

$$f(v, w) = v^tSw$$

So, the condition $f(x(v), w) + f(v, x(w)) = 0$ becomes:

$$x^tS + Sx = 0$$

For the even dimensional *orthogonal algebra* $\mathfrak{so}(2n, F)$ we take the nondegenerate symmetric bilinear form given by the matrix

$$S = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}$$

Thus $\mathfrak{so}(2n, F)$ is the set of all $2n \times 2n$ matrices

$$x = \begin{bmatrix} m & p \\ q & -m^t \end{bmatrix}$$

where $p = -p^t$ and $q = -q^t$. The Cartan subalgebra H is the set of diagonal matrices and H^* is generated by the n functions ϵ_i where $\epsilon_i(h) = h_i$ is the ith diagonal entry of h. The off-diagonal entries of the matrix m are coefficients of the matrix x_{ij} and

$$[h, x_{ij}] = (h_i - h_j)x_{ij} = (\epsilon_i - \epsilon_j)(h)x_{ij}$$

So, x_{ij} is in L_α where $\alpha = \epsilon_i - \epsilon_j$ and L_α is 1-dimensional. Also, $x_{ji} \in L_{-\alpha}$ and $[x_{ij}, x_{ji}] = h_\alpha$ has $\alpha(h_\alpha) = 2$.

The ij entry of p is $p_{ij} = -p_{ji}$. Let $y_{ij}, 1 \le i < j \le n$ be the basis element corresponding to this entry. Then

$$[h, y_{ij}] = (h_i + h_j)y_{ij} = (\epsilon_i + \epsilon_j)(h)y_{ij}$$

Similarly, the ij entry of q is $q_{ij} = -q_{ji}$. If z_{ij} is the negative of the corresponding basis element (with $q_{ij} = -1$) then

$$[h, z_{ij}] = (-h_i - h_j)z_{ij} = (-\epsilon_i - \epsilon_j)(h)z_{ij}$$

So, $y_{ij} \in L_\beta$ and $z_{ij} \in L_{-\beta}$ where $\beta = \epsilon_i + \epsilon_j$. Also, $[y_{ij}, z_{ij}] = h_\beta$ where $\beta(h_\beta) = 2$.

We see that the set of roots seems to be of type D_n (but we did not yet show that the inner product is the same). A base for this is the set of roots:

$$\Delta = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \cdots, \epsilon_{n-1} - \epsilon_n, \epsilon_{n-1} + \epsilon_n\}$$

(Proof that this is a base) Adding consecutive simple roots from the first $n-1$ gives all roots of the form $\epsilon_i - \epsilon_j$. To get $\epsilon_i + \epsilon_j$ take

$$\epsilon_i - \epsilon_j + 2(\epsilon_j - \epsilon_{n-1}) + (\epsilon_{n-1} - \epsilon_n) + (\epsilon_{n-1} + \epsilon_n)$$

All other roots are negatives of these. So, Δ is a base.

We also have a root space decomposition

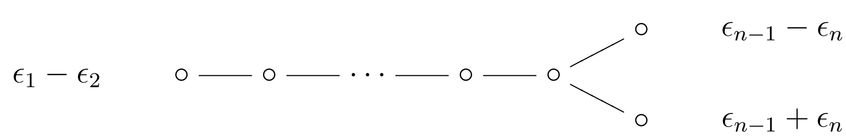
$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

And, each L_α is one-dimensional and the intersection of the kernels of $\alpha : H \to F$ is zero. These facts together imply the following.

Theorem 11.2.1. *$\mathfrak{so}(2n, F)$ is semisimple with root system of type D_n.*

Proof. To show that $\mathfrak{so}(2n, F)$ is semisimple we need to show that there is no nontrivial abelian ideal. So, suppose there is an abelian ideal J. Then H acts on J and, therefore, J decomposes into weight spaces for this action. But the weight space decomposition of J must be compatible with the decomposition of L. So J is a sum of $J_0 = J \cap H$ with $J_\beta = L_\beta$ for certain roots β. If any $J_\beta = L_\beta$ is nonzero, then J also contains $h_\beta \in [L_{-\beta} L_\beta] \subseteq [LJ] \subseteq J$. But h_β and L_β do not commute, contradicting the assumption that J is abelian. So, $J = J_0 \subseteq H$. If $h \in J \neq 0$ there is some root α which is nonzero on h. Then $[h, x_\alpha] = \alpha(h) x_\alpha$ is a nonzero element of L_α which is a contradiction. So, $\mathfrak{so}(2n, F)$ is semisimple.

The Cartan matrix has entries $\alpha(h_\beta)$. But the calculation shows that each h_β is the dual of β with respect to the basis ϵ_i. So, this is equal to the dot product of α, β. Apply this to the base to get the usual Dynkin diagram for D_n since we have two copies of A_{n-1} differing only in the last simple roots which are perpendicular.



□

11.3. **Symplectic algebra $\mathfrak{sp}(2n, F)$.** For the next example we take f to be nondegenerate and skew symmetric:

$$f(v, v) = 0$$

This is called a *symplectic form* on V.

Proposition 11.3.1. *Given a symplectic form f on V, there is a basis $v_1, \cdots, v_n, w_1, \cdots, w_n$ for V so that $f(v_i, v_j) = 0 = f(w_i, w_j)$ for all i, j and $f(v_i, w_j) = \delta_{ij}$. (This is called a* symplectic basis *for V.)*

Proof. This is by induction on the dimension of V. If the dimension is 0 there is nothing to prove. If $\dim V > 0$ then choose any nonzero vector $v_1 \in V$. SInce f is nondegenerate, there is a $w_1 \in V$ so that $f(v_1, w_1) = 1$. Since $f(v_1, v_1) = 0$, v_1, w_1 are linearly independent and their span W is 2-dimensional. Let

$$W^\perp = \{x \in V \mid f(x, v_1) = f(x, w_1) = 0\}$$

Then $W \cap W^\perp = 0$ and $\dim W^\perp = \dim V - 2$. So, $V = W \oplus W^\perp$. Since $W^\perp$ is perpendicular to W, the restriction of f to $W^\perp$ is symplectic. So, $W^\perp$ has a basis $v_2, \cdots, v_n, w_2, \cdots, w_n$ with the desired properties. Together with v_1 and w_1 we get a symplectic basis for W. □

Definition 11.3.2. The *symplectic Lie algebra* $\mathfrak{sp}(2n, F)$ is defined to be the algebra of all $x \in \mathfrak{sl}(2n, F)$ so that

$$x^t \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} = - \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} x$$

In other words, $\mathfrak{sp}(2n, F)$ is the set of all $2n \times 2n$ matrices

$$x = \begin{bmatrix} m & p \\ q & -m^t \end{bmatrix}$$

where $p = p^t$ and $q = q^t$. Again we have H the diagonal matrices. H^* is n-dimensional with basis ϵ_i, off-diagonal entries of m give x_{ij} with

$$[h, x_{ij}] = \alpha(h) x_{ij}$$

where $\alpha = \epsilon_i - \epsilon_j$ and $[x_{ij}, x_{ji}] = h_\alpha$. The entries of p give y_{ij} with

$$[h, y_{ij}] = \beta(h) y_{ij}$$

where $\beta = \epsilon_i + \epsilon_j$ where now we include the case $i = j$. The entries of q give basis elements z_{ij} with

$$[h, z_{ij}] = -\beta(h) z_{ij}$$

and

$$[y_{ij}, z_{ij}] = h_\beta$$

Thus we have a root system of type C_n. Note that, when $i = j$, the element h_β is not exactly the dual of β since $\beta = 2\epsilon_i$ whereas h_b is dual to ϵ_i. So, the Cartan matrix $\alpha_i(h_{\alpha_j})$ is not symmetric in this case.

Theorem 11.3.3. *$\mathfrak{sp}(2n, F)$ is semisimple with root system C_n.*

Proof. This follows from the root space decomposition just as in the proof of Theorem 11.2.1. The base for this root system is

$$\Delta = \{\epsilon_1 - \epsilon_2, \cdots, \epsilon_{n-1} - \epsilon_n, 2\epsilon_n\}$$

with Dykin diagram

$$\epsilon_1 - \epsilon_2 \qquad \circ - \circ - \circ - \cdots - \circ - \circ \Leftarrow \circ \qquad 2\epsilon_n$$

(The double arrow points to the shorter root.) □

11.4. **Orthogonal algebra $\mathfrak{so}(2n+1, F)$.** The odd dimensional *orthogonal algebra* $\mathfrak{so}(2n+1, F)$ is the Lie algebra given by the nondegenerate symmetric bilinear form corresponding to the matrix

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{bmatrix}$$

Thus $\mathfrak{so}(2n+1, F)$ is the set of all $2n+1 \times 2n+1$ matrices x so that $x^t S + Sx = 0$. In other words

$$x = \begin{bmatrix} 0 & c^t & -b^t \\ b & m & p \\ -c & q & -m^t \end{bmatrix}$$

where $p = -p^t$ and $q = -q^t$. We number the rows and columns $0, 1, 2, \cdots, 2n$.

The Cartan subalgebra H is the set of diagonal matrices and H^* is generated by the n functions ϵ_i where $\epsilon_i(h) = h_i$ is the ith diagonal entry of h. As in the case of $\mathfrak{so}(2n, F)$, the off-diagonal entries of the matrix m are coefficients of the matrix x_{ij} and

$$[h, x_{ij}] = (h_i - h_j)x_{ij} = \alpha(h)x_{ij}$$

where $\alpha = \epsilon_i - \epsilon_j$. So, x_{ij} is in L_α and $x_{ji} \in L_{-\alpha}$ and $[x_{ij}, x_{ji}] = h_\alpha$ has $\alpha(h_\alpha) = 2$.

We also have basic matrices $y_{ij}, 1 \leq i < j \leq n$ for p and z_{ij} for q so that

$$[h, y_{ij}] = (h_i + h_j)y_{ij} = \beta(h)y_{ij}$$

where $\beta = \epsilon_i + \epsilon_j$ and

$$[h, z_{ij}] = -\beta(h)z_{ij}$$

Also, $[y_{ij}, z_{ij}] = h_\beta$ where $\beta(h_\beta) = 2$.

What is new is the basic matrices b_i, c_i for b, c with

$$[h, b_i] = h_i b_i = \gamma(h)b_i$$

where $\gamma = \epsilon_i$ and

$$[h, c_i] = -h_i c_i = -\gamma(h)c_i$$

But $h_\gamma = 2[b_i, c_i]$ since

$$\gamma([b_i, c_i]) = 1$$

The root system consists of $\pm\epsilon_i \pm \epsilon_j$ and $\pm\epsilon_i$. The basic roots are given by

$$\epsilon_1 - \epsilon_2, \cdots, \epsilon_{n-1} - \epsilon_n, \epsilon_n$$

To see that this forms a base for the root system, note that adding consecutive elements gives any $\epsilon_i - \epsilon_j$ and adding the terms from $\epsilon_j - \epsilon_{j+1}$ to ϵ_n gives ϵ_j. And $\epsilon_i - \epsilon_j + 2\epsilon_j = \epsilon_i + \epsilon_j$. So, this is a base. Since ϵ_n is a short root, the Dynkin diagram is:

$\epsilon_1 - \epsilon_2$ ○ —— ○ —— ○ —— ⋯ —— ○ —— ○ ⟹ ○ ϵ_n

and this is a root system of type B_n.

Exercise 11.4.1. Show that $\mathfrak{sl}(n+1, F)$ is a semisimple Lie algebra with root system A_n:

$\epsilon_1 - \epsilon_2$ ○ —— ○ —— ○ —— ⋯ —— ○ —— ○ $\epsilon_n - \epsilon_{n+1}$