11.4. **Orthogonal algebra $\mathfrak{so}(2n+1,F)$.** The odd dimensional *orthogonal algebra* $\mathfrak{so}(2n+1,F)$ is the Lie algebra given by the nondegenerate symmetric bilinear form corresponding to the matrix

$$S=\begin{bmatrix}1&0&0\\0&0&I_n\\0&I_n&0\end{bmatrix}$$

Thus $\mathfrak{so}(2n+1,F)$ is the set of all $2n+1\times 2n+1$ matrices x so that $x^tS+Sx=0$. In other words

$$x=\begin{bmatrix}0&c^t&-b^t\\b&m&p\\-c&q&-m^t\end{bmatrix}$$

where $p=-p^t$ and $q=-q^t$. We number the rows and columns $0,1,2,\cdots,2n$.

The Cartan subalgebra H is the set of diagonal matrices and H^* is generated by the n functions ϵ_i where $\epsilon_i(h)=h_i$ is the ith diagonal entry of h. As in the case of $\mathfrak{so}(2n,F)$, the off-diagonal entries of the matrix m are coefficients of the matrix x_{ij} and

$$[h,x_{ij}]=(h_i-h_j)x_{ij}=\alpha(h)x_{ij}$$

where $\alpha=\epsilon_i-\epsilon_j$. So, x_{ij} is in L_α and $x_{ji}\in L_{-\alpha}$ and $[x_{ij},x_{ji}]=h_\alpha$ has $\alpha(h_\alpha)=2$.

We also have basic matrices $y_{ij}, 1\le i<j\le n$ for p and z_{ij} for q so that

$$[h,y_{ij}]=(h_i+h_j)y_{ij}=\beta(h)y_{ij}$$

where $\beta=\epsilon_i+\epsilon_j$ and

$$[h,z_{ij}]=-\beta(h)z_{ij}$$

Also, $[y_{ij},z_{ij}]=h_\beta$ where $\beta(h_\beta)=2$.

What is new is the basic matrices b_i,c_i for b,c with

$$[h,b_i]=h_ib_i=\gamma(h)b_i$$

where $\gamma=\epsilon_i$ and

$$[h,c_i]=-h_ic_i=-\gamma(h)c_i$$

But $h_\gamma=2[b_i,c_i]$ since

$$\gamma([b_i,c_i])=1$$

The root system consists of $\pm\epsilon_i\pm\epsilon_j$ and $\pm\epsilon_i$. The basic roots are given by

$$\epsilon_1-\epsilon_2,\cdots,\epsilon_{n-1}-\epsilon_n,\epsilon_n$$

To see that this forms a base for the root system, note that adding consecutive elements gives any $\epsilon_i-\epsilon_j$ and adding the terms from $\epsilon_j-\epsilon_{j+1}$ to ϵ_n gives ϵ_j. And $\epsilon_i-\epsilon_j+2\epsilon_j=\epsilon_i+\epsilon_j$. So, this is a base. Since ϵ_n is a short root, the Dynkin diagram is:

$\epsilon_1-\epsilon_2$ ∘ — ∘ — ∘ — ⋯ — ∘ — ∘ ⇒ ∘ ϵ_n

and this is a root system of type B_n.

Exercise 11.4.1. Show that $\mathfrak{sl}(n+1,F)$ is a semisimple Lie algebra with root system A_n:

$\epsilon_1-\epsilon_2$ ∘ — ∘ — ∘ — ⋯ — ∘ — ∘ $\epsilon_n-\epsilon_{n+1}$

12. Exceptional Lie algebras and automorphisms

We have constructed four infinite families of semisimple algebras:

(1) $\mathfrak{sl}(n+1,F)$ has type A_n
(2) $\mathfrak{so}(2n,F)$ has type D_n
(3) $\mathfrak{so}(2n+1,F)$ has type B_n
(4) $sp(2n,F)$ has type C_n.

Theorem 12.0.2. *If L is a simple Lie algebra over an algebraically closed field of characteristic zero then L either belongs to one of the four infinite families listed above or it is isomorphic to one of the five exceptional Lie algebras of type E_6, E_7, E_8, F_4, G_2.*

We will construct a Lie algebra of type G_2 later. For now we will just construct the root systems of type E_8 and F_4.

12.1. E_6, E_7, E_8**.** The root systems E_6, E_7 can be constructed from the root system (E,Φ) of type E_8 as follows. Let Δ' be the subset of the base Δ corresponding to the subdiagram of E_8 corresponding to E_6 or E_7. Let E' be the span of Δ' in E. Let $\Phi' = \Phi \cap E'$. Then (E/Φ') will be a root system of type E_6 or E_7. Therefore, it suffices to construct a root system of type E_8.

Let $E = \mathbb{R}^8$. The root system is:

$$\Phi = \{\pm\epsilon_i \pm \epsilon_j\} \cup \left\{ \frac{1}{2}\sum_{i=1}^{8} \pm\epsilon_i \text{ with even number of } - \text{ signs} \right\}$$

Note that the first part $\{\pm\epsilon_i \pm \epsilon_j\}$ is the root system of type D_8. Also, all of these roots have the same length. A base for this root system is

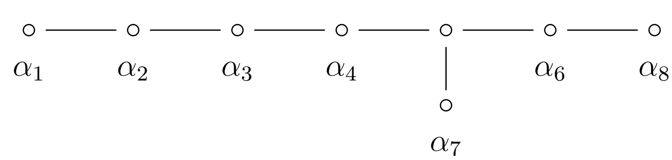
$$\epsilon_2 - \epsilon_3, \cdots, \epsilon_7 - \epsilon_8, \epsilon_7 + \epsilon_8, \frac{1}{2}\left(\epsilon_1 + \epsilon_8 - \sum_{i=2}^{7} \epsilon_i\right)$$

We show that this is a base:

Since the first 7 roots form a base for D_7 we know that any root of the form $\epsilon_i \pm \epsilon_j$ for $2 \le i < j \le n$ is a positive linear combination of the first 7 roots. By adding twice the last simple root α_8, we can get $\epsilon_1 - \epsilon_2$:

$$\epsilon_1 - \epsilon_2 = 2\alpha_8 + (\epsilon_3 + \epsilon_4) + (\epsilon_5 + \epsilon_6) + (\epsilon_7 - \epsilon_8)$$

with a single α_8 plus roots of the form $\epsilon_i \pm \epsilon_j$ we get any sum $\frac{1}{2}\sum \pm\epsilon_i$ with the sign of ϵ_1 being positive (with an even number of - signs). Note that α_8 is perpendicular to every other simple root except for $\epsilon_7 - \epsilon_8 = \alpha_6$. So, the Dynkin diagram is:



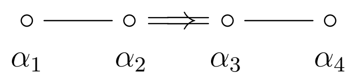
12.2. F_4. This root system is given by $E = \mathbb{R}^4$ and

$$\Phi = \{\pm\epsilon_i\} \cup \{\pm\epsilon_i \pm \epsilon_j\} \cup \{\tfrac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)\}$$

where these roots have lengths $1, \sqrt{2}, 1$ resp. The base is given by:

$$\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_3, \tfrac{1}{2}(-\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4)$$

The last two roots are smaller so the Dynkin diagram is:



12.3. **Automorphisms of Φ.** (preparing)