12. Exceptional Lie algebras and automorphisms

We have constructed four infinite families of semisimple algebras:

(1) $\mathfrak{sl}(n+1, F)$ has type A_n
(2) $\mathfrak{so}(2n, F)$ has type D_n
(3) $\mathfrak{so}(2n+1, F)$ has type B_n
(4) $sp(2n, F)$ has type C_n.

Theorem 12.0.2. *If L is a simple Lie algebra over an algebraically closed field of characteristic zero then L either belongs to one of the four infinite families listed above or it is isomorphic to one of the five exceptional Lie algebras of type E_6, E_7, E_8, F_4, G_2.*

We will construct a Lie algebra of type G_2 later. For now we will just construct the root systems of type E_8 and F_4.

12.1. E_6, E_7, E_8**.** The root systems E_6, E_7 can be constructed from the root system (E, Φ) of type E_8 as follows. Let Δ' be the subset of the base Δ corresponding to the subdiagram of E_8 corresponding to E_6 or E_7. Let E' be the span of Δ' in E. Let $\Phi' = \Phi \cap E'$. Then (E/Φ') will be a root system of type E_6 or E_7. Therefore, it suffices to construct a root system of type E_8.

Let $E = \mathbb{R}^8$. The root system is:

$$\Phi = \{\pm\epsilon_i \pm \epsilon_j\} \cup \left\{ \frac{1}{2} \sum_{i=1}^{8} \pm\epsilon_i \text{ with even number of } - \text{ signs} \right\}$$

Note that the first part $\{\pm\epsilon_i \pm \epsilon_j\}$ is the root system of type D_8. Also, all of these roots have the same length. A base for this root system is

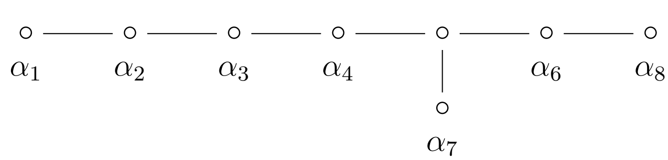
$$\epsilon_2 - \epsilon_3, \cdots, \epsilon_7 - \epsilon_8, \epsilon_7 + \epsilon_8, \frac{1}{2}\left(\epsilon_1 + \epsilon_8 - \sum_{i=2}^{7} \epsilon_i\right)$$

We show that this is a base:

Since the first 7 roots form a base for D_7 we know that any root of the form $\epsilon_i \pm \epsilon_j$ for $2 \le i < j \le n$ is a positive linear combination of the first 7 roots. By adding twice the last simple root α_8, we can get $\epsilon_1 - \epsilon_2$:

$$\epsilon_1 - \epsilon_2 = 2\alpha_8 + (\epsilon_3 + \epsilon_4) + (\epsilon_5 + \epsilon_6) + (\epsilon_7 - \epsilon_8)$$

with a single α_8 plus roots of the form $\epsilon_i \pm \epsilon_j$ we get any sum $\frac{1}{2}\sum \pm\epsilon_i$ with the sign of ϵ_1 being positive (with an even number of - signs). Note that α_8 is perpendicular to every other simple root except for $\epsilon_7 - \epsilon_8 = \alpha_6$. So, the Dynkin diagram is:



12.2. F_4**.** This root system is given by $E = \mathbb{R}^4$ and

$$\Phi = \{\pm\epsilon_i\} \cup \{\pm\epsilon_i \pm \epsilon_j\} \cup \{\tfrac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)\}$$

where these roots have lengths $1, \sqrt{2}, 1$ resp. The base is given by:

$$\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_3, \tfrac{1}{2}(-\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4)$$

The last two roots are smaller so the Dynkin diagram is:

$$\begin{array}{ccccccc} \circ & \text{———} & \circ & \Rrightarrow & \circ & \text{———} & \circ \\ \alpha_1 & & \alpha_2 & & \alpha_3 & & \alpha_4 \end{array}$$

12.3. **Automorphisms of Φ.**

Definition 12.3.1.

$$\mathrm{Aut}(\Phi) := \{f \in O(E) \mid f(\Phi) = \Phi\}$$

The *orthogonal group* $O(E)$ is the group of all linear isometries of E, i.e., f preserves the inner product, equivalently, f preserves lengths and angles.

Lemma 12.3.2. *The Weyl group W is a normal subgroup of* $\mathrm{Aut}(\Phi)$*.*

Proof. Clearly, W is a subgroup of $\mathrm{Aut}(\Phi)$ since it sends roots to roots and is generated by reflections. It remains to show that W is a normal subgroup.

W is generated by the reflections σ_β where $\beta \in \Phi$. For any $\tau \in \mathrm{Aut}(\Phi)$ we have:

$$\tau\sigma_\beta\tau^{-1} = \sigma_{\tau(\beta)} \in W$$

So, W is normal in $\mathrm{Aut}(\Phi)$. The equation follows from the fact that $\tau\sigma_\beta\tau^{-1}$ sends $\tau(\beta)$ to $-\tau(\beta)$ and also fixes every $x \perp \tau(\beta)$. □

Theorem 12.3.3. $\mathrm{Aut}(\Phi) \cong W \rtimes \Gamma$ *where Γ is the group of all $\tau \in \mathrm{Aut}(\Phi)$ so that $\tau(\Delta) = \Delta$.*

Proof. To prove that $\mathrm{Aut}(\Phi)$ is this semi-direct product it suffices to show that $W \cap \Gamma = \{1\}$ and that $W\Gamma = \mathrm{Aut}(\Phi)$.

To prove the first statement we recall that W acts simply transitively on the set of all bases. So, the only element of W which fixes Δ is the identity. To prove the second statement, take any $\tau \in \mathrm{Aut}(\Phi)$. Then $\tau(\Delta) = \Delta'$ is another base for Φ. So, there is some $w \in W$ so that $w(\Delta) = \Delta'$. Then $w^{-1}\tau \in \Gamma$ and $\tau = w(w^{-1}\tau) \in W\Gamma$. So, $\mathrm{Aut}(\Phi) = W \rtimes \Gamma$. □

Theorem 12.3.4. Γ *is the group of automorphisms of the Dynkin diagram.*

Before proving this, I pointed out that the automorphisms of the diagram are very easy to compute.

(1) For $A_n, n \geq 2$, $\Gamma = \mathbb{Z}/2$.
(2) For $D_n, n \geq 5$, $\Gamma = \mathbb{Z}/2$.
(3) For D_4, $\Gamma = S_3$, the symmetric group on 3 letters.
(4) For E_6, $\Gamma = \mathbb{Z}/2$.
(5) For all other Dynkin diagrams, Γ is trivial.

Proof. Take any $\tau \in \Gamma$. Then $\tau(\Delta) = \Delta$ means that τ permutes the elements of Δ and preserves lengths and angles. Therefore, τ gives an automorphism of the Dynkin diagram.

Conversely, suppose that τ is an automorphism of the Dynkin diagram. Then τ sends basic roots to basic roots. Since the basic roots form a basis for E, τ gives a linear isometry of E. The Dynkin diagram contains the information of which simple roots are long and which are short. So, τ sends long/short simple roots to long/short simple roots and also preserves the angles. So, τ is an isometry of E.

It remains to show that τ sends roots to roots. (I will finish this later.) $\square$