14. Isomorphism Theorem

This section contain the important theorem that two simple Lie algebras with the same Dynkin diagram are isomorphic. The proof uses the existence of a unique maximal root (Exercise 10.2.5). It also uses another lemma which we skipped:

Lemma 14.0.5. *Suppose that β is a positive root which is not simple. Then there exists a simple root α so that $\beta - \alpha$ is a positive root.*

Proof. Let $\beta = \sum k_i \alpha_i$. Then $(\beta, \beta) = \sum k_i(\beta, \alpha_i) > 0$ implies that $(\beta, \alpha_i) > 0$ for some α_i. But this implies that $\beta - \alpha_i$ is a root (Lemma 9.4.5). And it must be a positive root since one of the other coefficients $k_j > 0$. □

14.1. Proof of the isomorphism theorem.

Proposition 14.1.1. *A semisimple Lie algebra L is generated by $L_\alpha, L_{-\alpha}$ for all $\alpha \in \Delta$.*

Proof. Consider the subalgebra L^+ generated by L_α for all $\alpha \in \Delta$. We claim that L^+ contains all L_β for $\beta \in \Phi_+$. If not then take the smallest positive root $\beta = \sum k_i \alpha_i$ so that L_β is not contained in L^+. Then there is some simple root α so that $\gamma = \beta - \alpha$ is a positive root (which is smaller than β). But then $L_\gamma \subseteq L^+$ and $[L_\alpha, L_\gamma] = L_\beta \subseteq L^+$.

Similarly, the subalgebra L^- generated by all $L_{-\alpha}$ for simple α contains all L_β for $\beta \in \Phi_-$. So the subalgebra L' of L generated by all $L_\alpha, L_{-\alpha}$ contains L_β for all $\beta \in \Phi$. On the other hand $h_\alpha \in [L_\alpha, L_{-\alpha}]$. So, L' also contains all h_α. But the h_α generate H since the α are a basis for H^*. So, $L' = L$. □

Theorem 14.1.2. *Let L, L' be simple Lie algebras over F. Let H, H' be Cartan subalgebras for L, L'. Let Φ, Φ' be the corresponding root systems. Suppose that $\Phi \cong \Phi'$. Choose any isomorphism $\Phi \cong \Phi'$. Let Δ, Δ' be corresponding bases. Let $\pi : H \to H'$ be the corresponding isomorphism (whose dual $\pi^* : H'^* \to H^*$ sends Δ' to Δ). Choose any nonzero $x_\alpha \in L_\alpha, x'_\alpha \in L'_\alpha$. Then there is a unique isomorphism $\overline{\pi} : L \to L'$ sending x_α to x'_α so that $\overline{\pi}|H = \pi$.*

In short: if $\Phi \cong \Phi'$ then $L \cong L'$.

Proof. (Uniqueness) First note there are unique elements $y_\alpha \in L_{-\alpha}$ and $y'_\alpha \in L'_{-\alpha}$ so that $[x_\alpha, y_\alpha] = h_\alpha$ and $[x'_\alpha, y'_\alpha] = h'_\alpha$. Since $\pi : H \to H'$ sends h_α to h'_α, $\overline{\pi}$ must send y_α to y'_α. But the elements x_α, y_α generate L by the proposition above. So, $\overline{\pi}$ is uniquely determined.

Next we consider the algebra $L \oplus L'$. This is a semisimple Lie algebra with exactly two nonzero proper ideal: L, L'. Take the "diagonal" D which is the subalgebra of $L \oplus L'$ generated by the elements $\overline{x}_\alpha = (x_\alpha, x'_\alpha)$ and $\overline{y}_\alpha = (y_\alpha, y'_\alpha)$. Then the projection map $L \oplus L' \to L$ sends D onto L and similarly projection to the second factor sends D onto L'. We will show that both of these projection maps are isomorphisms giving $L \cong D \cong L'$.

Let D^- be the subalgebra of $L \oplus L'$ generated by the elements $\overline{y}_\alpha$ for $\alpha \in \Delta$. Let $\beta \in \Phi_+$ be the unique maximal root and choose nonzero elements $z \in L_\beta, z' \in L'_\beta$. Let M be the D^- submodule of $L \otimes L'$ generated by the element $\overline{z} = (z, z') \in L_\beta \oplus L'_\beta$. In other words, M is the span of $\overline{z}$ and all elements of the form

$$[\overline{y}_{\alpha_1}[\overline{y}_{\alpha_2} \cdots [\overline{y}_{\alpha_k}\overline{z}] \cdots]] \tag{14.1}$$

Claim 1 $M \cap (L_\beta \oplus L'_\beta)$ is one-dimensional.

Pf: The generators $\overline{y}_\alpha$ of D^- sends $L_\beta \oplus L'_\beta$ into $L_{\beta-\alpha} \oplus L'_{\beta-\alpha}$ and $L_\gamma \oplus L'_\gamma$ into $L_{\gamma-\alpha} \oplus L'_{\gamma-\alpha}$. So, we never come back to the maximal root β.

Claim 2 $[D, M] \subseteq M$.

Pf: We show by induction on k that ad$\overline{x}_\alpha$ sends the expression 14.1 above into M.

The generators $\overline{x}_\alpha$ of D act trivially on $(z, z') \in L_\beta \oplus L'_\beta$. So, this statement is true for $k = 0$. Since the difference of two simple roots is never a root, $\overline{x}_\alpha$ commutes with $\overline{y}_{\alpha'}$ if $\alpha' \neq \alpha$. So, if $\alpha_1 \neq \alpha$ then

$$[\overline{x}_\alpha[\overline{y}_{\alpha_1}[\overline{y}_{\alpha_2} \cdots [\overline{y}_{\alpha_k}\overline{z}] \cdots]]] = [\overline{y}_{\alpha_1}[\overline{x}_\alpha[\overline{y}_{\alpha_2} \cdots [\overline{y}_{\alpha_k}\overline{z}] \cdots]]]$$

lies in $[\overline{y}_{\alpha_1} M] \subseteq M$ by induction on k. Therefore, we may assume that $\alpha_1 = \alpha$. Then

$$[\overline{x}_\alpha[\overline{y}_\alpha[\overline{y}_{\alpha_2} \cdots [\overline{y}_{\alpha_k}\overline{z}] \cdots]]] = [\overline{y}_\alpha[\overline{x}_\alpha[\overline{y}_{\alpha_2} \cdots [\overline{y}_{\alpha_k}\overline{z}] \cdots]]] + [\overline{h}_\alpha[\overline{y}_{\alpha_2} \cdots [\overline{y}_{\alpha_k}\overline{z}] \cdots]]]$$

where $\overline{h}_\alpha = (h_\alpha, h'_\alpha) = [\overline{x}_\alpha, \overline{y}_\alpha]$. The first summand lies in M by induction. The second summand also lies in M since ad$\,\overline{h}_\alpha$ acts by multiplication by the same scalar in both coordinates of $L_\gamma \oplus L\gamma'$.

Claim 3 $D \neq L \oplus L'$.

Pf: Otherwise, M is an ideal by Claim 2 and proper by Claim 1. This is a contradiction since the only nontrivial proper ideals are $L \oplus 0$ and $0 \oplus L'$.

Claim 4 $D \cap L = 0 = D \cap L'$.

Pf: Suppose that $D \cap L$ is nonzero. Then it contains some $(w, 0)$ where $w \in L$ is nonzero. But $\overline{x}_\alpha \in D$ acts on $(w, 0)$ by $[\overline{x}_\alpha, (w, 0)] = ([x_\alpha w], 0)$ and similarly for $\overline{y}_\alpha$. By the Proposition, x_α, y_α generate L. So, $[D, (w, 0)] = ([L, w], 0) = (L, 0)$ which would imply that $D = L \oplus L'$ contradicting Claim 3. Similarly $D \cap L' = 0$.

This implies that the projection maps $D \to L, D \to L'$ are isomorphisms as claimed. These isomorphisms send $\overline{x}_\alpha$ to x_α, x'_α and similarly for $\overline{y}_\alpha$. But then the composition $L \to D \to L'$ sends x_α to x'_α as claimed. □

14.2. **Automorphisms of L.** The isomorphism theorem can also be used when $L = L'$. In that case it says that any automorphism of Φ extends to an automorphism of the Lie algebra L.

Corollary 14.2.1. *We have an epimorphism of groups:* $\text{Aut}(L) \twoheadrightarrow \text{Aut}(\Phi)$.

There is one automorphism in particular that we can write down:

Corollary 14.2.2. *If L is a simple Lie algebra and $x_\alpha \in L_\alpha, y_\alpha \in L_{-\alpha}$ so that $[x_\alpha, y_\alpha] = h_\alpha$ then there is an automorphism σ of L so that $\sigma(x_\alpha) = -y_\alpha$, $\sigma(y_\alpha) = -x_\alpha$ and $\sigma(h_\alpha) = -h_\alpha$. (Consequently, $\sigma(h) = -h$ for all $h \in H$ and $\sigma^2 = id_L$.)*

Exercise 14.2.3. (1) Show that Aut L contains a subgroup isomorphic to Γ.
(2) Prove that Φ has a unique maximal root. (See Exercise 10.2.5.)
(3) Complete the proof of Theorem 12.3.4.