15. Cartan subalgebras

The next project is to show that the Cartan subalgebra H of a semisimple Lie algebra L is unique up to an automorphism of L. However, it turns out to be easier to generalize the notion of Cartan subalgebra to a not necessarily semisimple Lie algebra and then show that this more general notion is unique up to an inner automorphism of L.

There are two equivalent definitions for a Cartan subalgebra H of a general Lie algebra L:

(1) H is a self-normalizing nilpotent subalgebra, i.e., a nilpotent subalgebra of L so that
$$H = N_L(H) := \{x \in L \mid [xH] \subseteq H\}$$
For L semisimple, this is equivalent to the theorem that $H = L_0$.

(2) H is a minimal Engel subalgebra (defined below).

Humphreys points out that the ground field F can be of arbitrary characteristic. But, when F is finite, we need to assume that F has more than $2 \dim L$ elements.

15.1. Engel subalgebras.

For any endomorphism x of $V \cong F^n$ let
$$V_0(x) := \ker(x^n) = \ker(x^m) \quad \forall m \geq n.$$
Note that x acts nilpotently on $V_0(x)$ and x acts as an automorphism on $V/V_0(x)$.

Lemma 15.1.1. *Suppose that $z, x \in \mathfrak{gl}(V)$ and $V \cong F^n$.*

(1) *The set of all $c \in F$ so that $z + cx$ is nilpotent is either finite with $\leq n$ elements or all of F.*

(2) *The set of all $c \in F$ so that $z + cx$ is not an automorphism of V is either finite with $\leq n$ elements or all of F.*

Proof. The characteristic polynomial of $z + cx$,
$$p(T) = \det(z + cx - T)$$
is a polynomial in T of degree n whose coefficients are polynomials in the entries of the matrix for $z + cx$. Considering z, x to be constant, these coefficients are polynomials in c:
$$p(T) = (-1)^n T^n + g_1(c)T^{n-1} + \cdots + g_n(c)$$
where $\deg g_i(c) \leq i \leq n$.

(2) $z + cx$ is a not an automorphism iff its determinant is zero: $p(0) = g_n(c) = 0$. But g_n has at most n roots unless it is the zero polynomial in which case $g_n(c) = 0$ for all $c \in F$.

(1) $z + cx$ is nilpotent iff $p(T) = (-1)^n T^n$. This happens iff $g_i(c) = 0$ for all i. This is an intersection of sets each of which is either finite with $\leq n$ elements or all of F. □

Exercise 15.1.2. Show that $L_0(\operatorname{ad} x)$ is a subalgebra of L and $x \in L_0(\operatorname{ad} x)$. Show that a subalgebra K of L is nilpotent iff $K \subseteq L_0(\operatorname{ad} x)$ for all $x \in K$.

Definition 15.1.3. An *Engel subalgebra* of L is any subalgebra of the form $E = L_0(\operatorname{ad} x)$ for some $x \in L$. We call x an *annihilator* for E.

Lemma 15.1.4. *Let K any subalgebra of L and let $z \in K$ be so that the Engel subalgebra $E = L_0(\operatorname{ad} z)$ is minimal among all those with annihilator in K. Suppose that $K \subseteq E$. Then $E \subseteq L_0(\operatorname{ad} x)$ for all $x \in K$. In particular, K is nilpotent.*

Recall that we are assuming F has at least $2n + 1$ elements where $n = \dim_F L$.

Proof. Take any $x \in K$. Then, for any $c \in F$, $z + cx \in K \subseteq E$. So, $\operatorname{ad}(z + cx)$ stabilizes E and induces an endomorphism of both E and L/E. Since $\operatorname{ad} z$ is an isomorphism on L/E, $\operatorname{ad}(z + cx)$ will be an isomorphism on L/E for all but a finite number ($\leq n$) of $c \in F$. Since F has at least $2n + 1$ elements, $\operatorname{ad}(z + cx)$ is an isomorphism on L/E for at least $n + 1$ values of c. For each of these values of c we must have

$$L_0(\operatorname{ad}(z + cx)) \subseteq E$$

But, the minimality of E implies that these must be equal. Therefore, $\operatorname{ad}_E(z + cx)$ is nilpotent for $n+1$ values of c. By the previous lemma, $\operatorname{ad}_E(z+cx)$ must be nilpotent for all values of $c \in F$. This implies that $L_0(\operatorname{ad}(z + x)) \supseteq E$ for all $x \in K$. Replacing x with $x - z$ gives the result. □

Lemma 15.1.5. *Any subalgebra K of L which contains an Engel subalgebra is self-normalizing, i.e., $K = N_L(K)$.*

Proof. Suppose that K contains $E = L_0(\operatorname{ad} x)$. Then $x \in E \subseteq K$. So, $[x, K] \subseteq K$ and $[x, N_L(K)] \subseteq K$. So, $\operatorname{ad} x$ stabilizes E, K and $N_L(K)$. But we know that $\operatorname{ad} x$ acts as an automorphism on L/E and therefore on L/K. And $\operatorname{ad} x$ annihilates $N_L(K)/K$. Therefore, $N_L(K)/K = 0$. So, $N_L(K) = K$. □

15.2. Cartan subalgebras.

Theorem 15.2.1. *Let H be a subalgebra of L. Then tfae.*

(1) *H is a minimal Engel subalgebra of L.*
(2) *H is nilpotent and self-normalizing.*

Proof. (1) ⇒ (2) If $H = L_0(\operatorname{ad} x)$ is minimal then $x \in H$ and H is nilpotent by Lemma 15.1.4. By Lemma 15.1.5, any Engel subalgebra is self-normalizing.

(2) ⇒ (1) Since H is nilpotent, $H \subseteq L_0(\operatorname{ad} x)$ for all $x \in H$. In particular, H cannot properly contain any Engel subalgebra. Therefore, it suffices to show that $H = L_0(\operatorname{ad} x)$ for some $x \in H$.

Suppose not. Let $E = L_0(\operatorname{ad} z)$ be minimal. Then, $E \subseteq L_0(\operatorname{ad} x)$ for all $x \in H$ by Lemma 15.1.4. This implies that every element of H acts nilpotently on E/H. By Engel's Lemma 3.1.3, there is some nonzero element $x + H \in E/H$ which is annihilated by every element of H. In other words $[H, x + H] \subseteq H$. So, $[H, x] \subseteq x$ which means $x \in N_L(H) = H$. This is a contradiction. □

Corollary 15.2.2. *If H is a subalgebra of a semisimple Lie algebra L then tfae.*

(2) *H is nilpotent and self-normalizing.*
(3) *H is a Cartan subalgebra of L, i.e., a maximal subalgebra which is abelian in which all the elements are semisimple.*

Proof. We already observed that (3) $\Rightarrow$ (2) since abelian implies nilpotent and $H = L_0$ implies $H = N_L(H)$. To show the converse, suppose that H is a minimal Engel subalgebra of L. Then $H = L_0(\operatorname{ad} x)$. But $x = x_s + x_n$ and x_n is nilpotent on all of L. So, $L_0(\operatorname{ad} x) = L_0(\operatorname{ad} x_s)$. But, for semisimple elements, generalized eigenspaces are the same as eigenspaces. So,

$$H = L_0(\operatorname{ad} x_s) = \ker(\operatorname{ad} x_s) = C_L(x_s)$$

Since Fx_s is abelian and its elements are all semisimple, Fx_s is contained in some Cartan subalgebra C of L. Since C is abelian it is contained in $H = C_L(x_s)$. But we just proved that Cartan subalgebras are Engel subalgebras. Therefore C is an Engel subalgebra contained in the minimal Engel subalgebra H. Therefore $C = H$, making H a Cartan subalgebra. □

Definition 15.2.3. A *Cartan subalgebra* (CSA) of a Lie algebra L is defined to be any nilpotent self-normalizing subalgebra of L.

By the corollary above, this definition agrees with the previous definition when L is semisimple. The proof of the corollary also gives the following.

Corollary 15.2.4. *For any Cartan subalgebra H of a semisimple Lie algebra L, there exists an element $x \in H$ so that $H = C_L(x)$.*

We call x a *regular semisimple element* of L.

15.3. **Functorial properties of CSA's.**

Lemma 15.3.1. *Any epimorphism of Lie algebras $\varphi : L \to L'$ takes CSA's of L onto CSA's of L'.*

Proof. Let H be a CSA of L. Then $\varphi(H)$ is nilpotent. So, it suffices to show that $\varphi(H)$ is self-normalizing. But $\varphi(x)$ normalizes $\varphi(H)$ iff x normalizes $H + \ker\varphi$ in L. But $H + \ker\varphi$ contains the Engel subalgebra H and is thus self-normalizing by Lemma 15.1.5. Therefore, $x \in H + \ker\varphi$ making $\varphi(x) \in \varphi(H)$. So, $\varphi(H)$ is self-normalizing and nilpotent. □

Proposition 15.3.2. *For any epimorphism $\varphi : L \to L'$ and any CSA H' of L', let $K = \varphi^{-1}(H')$. Then any CSA of K is a CSA of L.*

Proof. Let H be a CSA of K. Then H is nilpotent and $H = N_K(H)$. So, it suffices to show that $H = N_L(H)$. Since $\varphi(K) = H'$, the lemma above implies that $\varphi(H) = H'$ (since H' is the only CSA of H'). Let $x \in N_L(H)$. Then $\varphi(x)$ normalizes $\varphi(H) = H'$. So, $\varphi(x) \in H'$ making $x \in K$. But $H = N_K(H)$. So, $x \in H$, proving that $N_L(H) = H$ as required. □

Exercise 15.3.3.
1. If H is a CSA of L then prove that H is a maximal nilpotent subalgebra of L.
2. Prove Lemma 15.1.4 using the assumption that F has at least $n+1$ elements where $n = \dim L$.