16. Conjugacy theorems

We can now prove that the Cartan subalgebra H of any Lie algebra L is unique up to an inner automorphism of L. However, the proof in the book is too complicated. I will present an alternate proof which is based on Tauvel and Yu "Lie Algebras and Algebraic Groups". They prove directly that CSA's are all conjugate and derive as a fairly easy consequence the fact that Borel subalgebras are all conjugate.

The proof of Tauvel and Yu uses algebraic geometry. In order not to go too far into that area I will assume that the ground field is the complex number $\mathbb{C}$. I will use the elementary fact that any Zariski open subset of $\mathbb{C}^n$ is path connected. I will also use the inverse function theorem from multivariable calculus.

Recall that, for any field F, a subset of F^n is *Zariski closed* if it is the common set of zeros of a set of polynomials. When $n = 1$, a Zariski closed set is either finite or all of F.

Proposition 16.0.4. *Any nonempty Zariski open subset U of $\mathbb{C}^n$ is path connected and dense in the usual Euclidean topology on $\mathbb{C}^n$.*

Proof. Take any two points x, y in U. Let L be the affine line connecting x, y. Topologically, L is a plane since $L \cong \mathbb{C} \cong \mathbb{R}^2$. The intersection with U is Zariski open in L and therefore the complement of a finite subset of L. But the complement of a finite subset of a plane is path connected.

Doing the same thing with $x \in U$ and $y \notin U$ we see that y is in the closure of U. So, U is dense in $\mathbb{C}^n$. □

16.1. Generic elements.

Now take any Lie algebra L. As a vector space $L \cong F^n$. For each element $x \in L$, take the characteristic polynomial $p_x(T)$:

$$p_x(T) = \det(\operatorname{ad} x - TI_n) = (-1)^n T^n + g_1(x)T^{n-1} + \cdots + g_n(x)$$

Each coefficient $g_i(x)$ is a polynomial in x. The last term $g_n(x) = 0$ since $\operatorname{ad} x$ is never an automorphism of L. (Why?) The first term $g_0(x) = (-1)^n$ is never zero.

Definition 16.1.1. The *rank* of L is defined to be the smallest integer r so that $g_{n-r}(x) \neq 0$ for some $x \in L$. The elements $x \in L$ for which $g_{n-r} \neq 0$ are called *generic* or *regular*.

- The generic elements clearly form a nonempty Zariski open subset of L.
- x is generic iff the multiplicity of 0 as an eigenvalue of $\operatorname{ad} x$ is r (the rank of L).
- For $y \in L$ not generic, the multiplicity of 0 as eigenvalue of $\operatorname{ad} y$ is greater than r.
- Any automorphism ψ of L sends generic elements to generic elements.

Lemma 16.1.2. *Suppose that $x \in L$ is generic. Then $L_0(\operatorname{ad} x)$ is the unique CSA of L which contains x and it dimension is equal to the rank r of L.*

Proof. By definition, $\dim L_0(\operatorname{ad} x) = r$ iff x is regular. If y is not regular then $\dim L_0(\operatorname{ad} y) > r$. Therefore, $H = L_0(\operatorname{ad} x)$ is minimal Engel. So, it is a CSA. If C is any other CSA containing x then C is nilpotent. So, $\operatorname{ad} x$ is nilpotent on C which means $C \subseteq H$. But H is minimal. So, $C = H$. □

Next we will show that every CSA $H \subseteq L$ contains a generic element. The idea is to deform H by an automorphism of L close to the identity map. Since generic elements for an open dense subset of L, a general deformation of H will contain a generic element. So, H must contain a generic element. To do this rigorously, we first need to define the automorphism groups that we want to use.

16.2. **The group $\mathcal{E}(L)$.** Assume that F is algebraically closed of characteristic 0. Then for any $x \in L$, L decomposes into a direct sum of the generalized eigenspaces of $\operatorname{ad} x$:

$$L = \bigoplus_{\lambda} L_\lambda(\operatorname{ad} x)$$

where

$$L_\lambda(\operatorname{ad} x) = \ker(\operatorname{ad} x - \lambda)^n$$

Definition 16.2.1. $x \in L$ is called *strongly ad-nilpotent* if $x \in L_\lambda(\operatorname{ad} y)$ for some $y \in L$ and $\lambda \in F$. The set of strongly ad-nilpotent elements of L is denoted $\mathcal{N}(L)$. The subgroup of $\operatorname{Aut}(L)$ generated by $\exp \operatorname{ad} x$ for all $x \in \mathcal{N}(L)$ is denoted $\mathcal{E}(L)$.

Exercise 16.2.2. If K is a subalgebra of L then show that $\mathcal{N}(K) \subseteq \mathcal{N}(L)$ and that, for any $x \in \mathcal{N}(K)$, $\operatorname{ad}_K(x)$ is the restriction to K of $\operatorname{ad}_L(x)$. (Let $\mathcal{E}(L, K)$ denote the subgroup of $\mathcal{E}(L)$ generated by $\exp \operatorname{ad} x$ for all $x \in \mathcal{N}(K)$.)

If $\varphi : L \to L'$ is an epimorphism and $y \in L$ then show that $\varphi(L_\lambda(\operatorname{ad} y)) = L'_\lambda(\operatorname{ad} \varphi(y))$. Conclude that $\varphi(\mathcal{N}(L)) = \mathcal{N}(L')$.

Lemma 16.2.3. *If $\varphi : L \to L'$ is an epimorphism and $\sigma' \in \mathcal{E}(L')$ then there is a $\sigma \in \mathcal{E}(L)$ so that $\varphi\sigma = \sigma'\varphi$. I.e., the following diagram commutes.*

$$\begin{array}{ccc} L & \xrightarrow{\varphi} & L' \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma'} \\ L & \xrightarrow{\varphi} & L' \end{array}$$

Proof. σ' is a product (composition) of automorphism of L' of the form $\exp \operatorname{ad} x'$ for some $x' \in \mathcal{N}$. By the exercise, $x' = \varphi(x)$ for some $x \in \mathcal{N}(L)$. Then

$$\exp \operatorname{ad}(\varphi(x))\varphi(y) = \varphi\left((\exp \operatorname{ad} x)(y)\right)$$

In other words, $\varphi(\exp \operatorname{ad} x) = (\exp \operatorname{ad} x')\varphi$. Let σ be the product of the liftings $\exp \operatorname{ad} x$ for each factor $\exp \operatorname{ad} x'$ of σ'. □

From now on, we assume that $F = \mathbb{C}$.

Lemma 16.2.4. *The derivative of* $\exp \operatorname{ad} ty = \exp(t \operatorname{ad} y)$ *at* $t = 0$ *is* $\operatorname{ad} y$.

Proof. Let $z = \operatorname{ad} y$. Then

$$\exp(tz) = id_L + tz + t^2 z/2 + \cdots$$

$$\frac{d}{dt}\exp(tz) = z + tz + t^2 z/2 + \cdots$$

which is equal to z at $t = 0$. □

The lemma says that, when $t \in \mathbb{C}$ is close to 0 ($|t|$ is small).

$$(\exp \operatorname{ad} ty)(x) \approx x + t[y, x] \qquad (= x - t[x, y])$$

16.3. **Conjugacy of CSA's.**

Lemma 16.3.1. *Every CSA H of L contains a generic element.* ($\Rightarrow \dim H = \text{rank } L$)

Proof. Since H is Engel, $H = L_0(\operatorname{ad} x)$ for some $x \in H$. The generalized eigenspace decomposition of L wrt $\operatorname{ad} x$ is:

$$H \oplus L_{\lambda_1} \oplus L_{\lambda_2} \oplus \cdots \oplus L_{\lambda_k}$$

with $H = L_0$ having $\dim H = k \geq r$ and suppose $\dim L_{\lambda_i} = m_i$. For each i, choose a Jordan canonical form for $\operatorname{ad} x$ acting on L_{λ_i}. This gives a basis of y_{ij} for each L_{λ_i} so that the matrix of $\operatorname{ad} x$ is $\lambda_i I_{m_i}$ plus an upper triangular matrix. Then each $y_{ij} \in \mathcal{N}(L)$ and $\sum m_i = n - k$.

Consider the function $\psi : H \times \mathbb{C}^{n-k} \to L$ given by

$$\psi(h,t) = \sigma_t(h) \text{ where } \sigma_t = \prod_{j=1}^{m} \exp \operatorname{ad} t_{ij} y_{ij} \in \mathcal{E}(L)$$

The function ψ is smooth (C^∞) since, in matrix form, every entry is a polynomial. (ψ is *algebraic*.)

Claim The derivative of ψ at $(x, 0)$ is nonsingular.

Pf: By the lemma, the partial derivative of $\psi(h,t)$ wrt t_{ij} at $(h,t) = (x,0)$ is

$$\operatorname{ad} y_{ij}(x) = -[x, y_{ij}] = -\lambda_i y_{ij} - (*) y_{i,j-1}$$

where $* = 0$ or 1. if $y_j \in L_{\lambda_i}$. Therefore the matrix of the derivative $D\psi_{(x,0)}$ wrt the basis $\{y_{ij}\}$ plus any basis for H is

$$D\psi_{(x,0)} = \begin{bmatrix} I_k & 0 \\ 0 & U \end{bmatrix}$$

where U is an upper triangular matrix with nonzero diagonal entries $-\lambda_i$.

By the inverse function theorem, the image of ψ contains an open (in the Euclidean topology) neighborhood U of the point $x \in L$. Since the set of generic elements is open and dense in L, this open neighborhood contains a generic point x_{gen}. By definition $x_{gen} = \psi(h,t)$ for some $(h,t) \in H \times \mathbb{C}^{n-k}$. This implies that $h \in H$ is generic. Furthermore, $x_{gen} = \sigma(h)$ for some $\sigma \in \mathcal{E}(L)$. ∎

Note that, at the last step, we also get that $\sigma(H) = L_0(x_{gen})$ since this is the unique CSA containing x_{gen}. This proves the following lemma.

Lemma 16.3.2. *For any generic x in L there is an open neighborhood U of x so that, for any other generic element $x' \in U$ there is a $\sigma \in \mathcal{E}(L)$ so that $\sigma(L_0(\operatorname{ad} x)) = L_0(\operatorname{ad} x')$.*

Theorem 16.3.3. *The group $\mathcal{E}(L)$ acts transitively on the set of CSA's of L.*

Proof. Take any two CSA's $H_0, H_1 \subseteq L$. By the first lemma above, each $H_i = L_0(\operatorname{ad} x_i)$ for x_0, x_1 generic. Choose a continuous path $x_t, 0 \leq t \leq 1$ from x_0 to x_1 in the set of generic elements. Let $H_t = L_0(\operatorname{ad} x_t)$ for each $t \in [0,1]$. By the second lemma above, there is an open neighborhood U_t of each x_t and therefore an open interval J_t around $t \in [0,1]$ so that for all $s \in J_t$, $H_s = \sigma(H_t)$ for some $\sigma \in \mathcal{E}(L)$. Since $[0,1]$ is compact, there is a finite covering by these intervals J_t. We conclude that $H_1 = \sigma H_0$ for some $\sigma \in \mathcal{E}(L)$. ∎