16.4. **Borel subalgebras.** The book uses Borel subalgebras to prove that CSA's are conjugate. We are doing this backwards. Given that CSA's are conjugate by the action of $\mathcal{E}(L)$, we will show that Borel subalgebras are all conjugate. This is Humphreys proof which we read backwards.

16.4.1. *definition and basic properties.* A *Borel subalgebra* B of L is defined to be a maximal solvable subalgebra. The first two basic properties are clear.

Lemma 16.4.1. *Every Borel subalgebra is self normalizing:* $B = N_L(B)$. □

Recall that $Rad(L)$ is the unique maximal solvable ideal of L and that $L/Rad(L)$ is semisimple.

Lemma 16.4.2. *Every Borel subalgebra B of L contains the solvable radical $Rad(L)$. And $B \leftrightarrow B/Rad(L)$ gives a bijection between the set of Borel subalgebras of L and those of $L/Rad(L)$.* □

This lemma reduces the problem to the case when L is semisimple. We will show the following.

Theorem 16.4.3. *For any two Borel subalgebras B, B' of a semisimple Lie algebra L, there is an automorphism $\sigma \in \mathcal{E}(L)$ so that $B' = \sigma(B)$.*

With the lemma and the functorial properties of $\mathcal{E}(L)$ we get the following.

Corollary 16.4.4. *For any two Borel subalgebras B, B' of a Lie algebra L, there is an automorphism $\sigma' \in \mathcal{E}(L)$ so that $B' = \sigma'(B)$.*

Proof. By the theorem, there is a $\sigma \in \mathcal{E}(L/Rad(L))$ so that $\sigma(B/Rad(L)) = B'/Rad(L)$. By the functorial properties of $\mathcal{E}$, $\sigma \in \mathcal{E}(L/Rad(L))$ lifts to an element $\sigma' \in \mathcal{E}(L)$ so that $\varphi\sigma' = \sigma\varphi$. So,

$$\varphi\sigma'(B) = \sigma'(B)/Rad(L) = \sigma(B/Rad(L)) = B'/Rad(L)$$

and therefore, $\sigma(B) = B'$. □

It remains to prove the theorem.

16.4.2. *standard Borel subalgebras.* The standard example of a Borel subalgebra is given as follows.

Lemma 16.4.5. *Let L be a semisimple Lie algebra H a CSA, with root system Φ and base Δ. Then*

$$B(\Delta) := H \oplus \bigoplus_{\beta\in\Phi_+} L_\beta$$

is a Borel subalgebra of L. These are called the Standard Borel subalgebras *of L. Conversely, any Borel subalgebra of L which contains H is standard.*

Exercise 16.4.6. Show that $N(\Delta) := \bigoplus_{\beta\in\Phi_+} L_\beta$ is a maximal nilpotent subalgebra of L.

Proof. It is clear that $B(\Delta)$ is solvable since $[B(\Delta, B(\Delta)] = N(\Delta)$ is nilpotent.

Conversely, we claim that any solvable subalgebra B of L which contains H is contained in $B(\Delta)$ for some base Δ for Φ. This will prove both statements in our lemma. To prove this, let B be solvable and $H \subseteq B$. Then B has a weight space decomposition wrt H:

$$B = H \oplus \bigoplus_{\beta \in S} L_\beta$$

for some set of roots $S \subseteq \Phi$. But S cannot contain contain both β and $-\beta$ since, otherwise, L would contain the semisimple Lie algebra $S_\beta = L_{-\beta} \oplus Fh_\beta \oplus L_\beta$ which is not possible since all subalgebras of solvable algebras are solvable. But this implies that $S \subseteq \Phi_+$ with respect to some base Δ and thus $B \subseteq B(\Delta)$. □

Lemma 16.4.7. *Given a fixed CSA H of a semisimple Lie algebra L, any two Borel subalgebras B containing H are conjugate by an element of $\mathcal{E}(L)$.*

Proof. By the previous lemma we know that each $B \supseteq H$ has the form $B(\Delta)$. We also know that the Weyl group W acts transitively on the set of bases Δ. But W is generated by reflections σ_α which lift to the elements of $\mathcal{E}(L)$ given by:

$$\tau_\alpha = \exp \operatorname{ad} x_\alpha \cdot \exp \operatorname{ad} (-y_\alpha) \cdot \exp \operatorname{ad} x_\alpha$$

Claim 1 $\tau_\alpha(h_\alpha) = -h_\alpha$

Pf: Since $[x_\alpha, h_\alpha] = -2x_\alpha$, $[y_\alpha, h_\alpha] = 2y_\alpha$ and $[y_\alpha, x_\alpha] = -h_\alpha$ we get:

$$\exp \operatorname{ad} x_\alpha(h_\alpha) = h_\alpha + [x_\alpha, h_\alpha] + \frac{1}{2}[x_\alpha[x_\alpha, h_\alpha]] + \cdots = h_\alpha - 2x_\alpha$$

$$\begin{aligned}\exp \operatorname{ad} (-y_\alpha)(h_\alpha - 2x_\alpha) &= (h_\alpha - 2x_\alpha) - [y_\alpha, (h_\alpha - 2x_\alpha)] + \frac{1}{2}[y_\alpha[y_\alpha, (h_\alpha - 2x_\alpha)]] + \cdots \\ &= (h_\alpha - 2x_\alpha) - (2y_\alpha + 2h_\alpha) + \frac{1}{2}(4y_\alpha) \\ &= -h_\alpha - 2x_\alpha\end{aligned}$$

$$\exp \operatorname{ad} x_\alpha(-h_\alpha - 2x_\alpha) = (-h_\alpha - 2x_\alpha) + 2x_\alpha = -h_\alpha$$

Claim 2 If $h \in K = \ker \alpha$ then $\tau_\alpha(h) = h$.

Pf: This follows from the fact that $[h, x_\alpha] = \alpha(h)x_\alpha = 0$ and similarly $[h, y_\alpha] = 0$.

Together, these two Claims imply that $\tau_\alpha^* : H^* \to H^*$ sends α to $-\alpha$ since $\alpha\tau_\alpha(h) = \alpha(h) = 0$ for $h \in K$ and $\alpha\tau_\alpha(h_\alpha) = -2h_\alpha$. For any $x \in H^*$ orthogonal to α we have $x(h_\alpha) = 0$. So, $x\tau_\alpha(h_\alpha) = 0$ and $x\tau_\alpha(h) = x(h)$ for any $h \in K$. So, $\tau_\alpha^*(x) = x$. In other words, τ_α^* is equal to the reflection σ_α as claimed. Therefore, the subgroup of $\mathcal{E}(L)$ generated by the τ_α acts transitively on the set of Borel subalgebras of L containing H. □

Since $\mathcal{E}(L)$ acts transitively on the set of CSA's of L we get the following.

Theorem 16.4.8. *$\mathcal{E}(L)$ acts transitively on the set of all standard Borel subalgebras of L.*

The only thing left to show is the following.

Theorem 16.4.9. *All Borel subalgebras of a semisimple Lie algebra L are standard.*

Proof. Let B be a Borel subalgebra of L. To show that B is standard, it suffices to show that B contains a CSA H of L. So, we may assume that B does not contain any CSA.

<u>Claim 1</u> B contains at least one nonzero semisimple element t.

Pf: We know that B is self-normalizing. So, it cannot be nilpotent. Otherwise, B would be a CSA. So, B contains a nonzero element x which is not nilpotent. Then $x_s \neq 0$. But $[x_s, B] \subseteq B$ since x_s normalizes any subalgebra which is normalized by x. So, $x_s \in N_L(B) = B$ as claimed.

<u>Claim 2</u> We may assume t is not in the center of B.

Since t is semisimple, it is contained in a CSA H of L and H is abelian. If t is central in B then $H, B \subset C = C_L(t) \subsetneq L$ are CSA and Borel respectively in C. By induction on the size of L, there is some $\sigma \in \mathcal{E}(C)$ so that $\sigma(H) \subseteq B$. But $\sigma(H)$ is also a CSA of L and we are done.

Since t is semisimple, B decomposes as eigenspaces of the action of $\operatorname{ad} t$:

$$B = B_0 \oplus \bigoplus B_\lambda$$

Since t is not central in B, one of the B_λ is nonzero, $\lambda \neq 0 \in F$. Take $x \neq 0$ in B_λ. Then x is nilpotent (since $[x, B_a] \subseteq B_{a+\lambda}$) and $[t, x] = \lambda x$. So, $[t_0, x] = x$ where $t_0 = \frac{1}{\lambda} t$.

Let H be a CSA of L which contains t. Then, using the root space decomposition of L wrt H, we get

$$x = x_0 + \sum_{\beta \in S} c_\beta x_\beta$$

where $x_0 \in H, x_\beta \in L_\beta$ and $c_\beta \neq 0 \in F$ for each β is a subset $S \subseteq \Phi$.

<u>Claim 3</u> $x_0 = 0$ and $S \subseteq \Phi_+$ with respect to some base Δ. Thus $x \in N(\Delta)$.

Pf: The calculation

$$[t_0, x] = x = \sum c_\beta \beta(t_0) x_\beta$$

implies that $x_0 = 0$ and $\beta(t_0) = 1$ for all $\beta \in S$. So, β and $-\beta$ cannot both be in S. Claim 3 follows.

This implies $A = B \cap B(\Delta)$ contains the nonzero nilpotent element $x \in N = B \cap N(\Delta)$. Choose Δ so that A is maximal and $N \neq 0$. (If $A = B$ we are done.) Since $N(\Delta)$ is an ideal in $B(\Delta)$, N is an ideal in A. So, $A \subset K = N_L(N) \subsetneq L$. ($L$ has no ideals.)

<u>Claim 4</u> A is properly contained in $B \cap K$ and in $B(\Delta) \cap K$.

Pf: Since $N \subseteq B$ is nilpotent, the action of N on B/A is nilpotent and there is a $y \in B$, $y \notin A$ so that $[N, y] \subseteq A$. But $[N, y] \subseteq [B, B]$ is nilpotent. So, we must have $[N, y] \subseteq N$. So, $y \in K \cap B$ but $y \neq A$. The other case is similar.

Let C, C' be Borel subalgebras of K which contain $B \cap K, B(\Delta) \cap K$ respectively. (See Figure 1 on p.85 in the book.) Then, by induction on $\dim L$, there is a $\sigma \in \mathcal{E}(K)$ sending C to C'. So, we may assume that $C = C'$. Let B' be a Borel subalgebra of L containing $C = C'$. Then, $B' \cap B(\Delta) \supseteq K \cap B(\Delta) \supsetneq A = B \cap B(\Delta)$. Therefore, by maximality of A, B' is standard. But we also have $B \cap B' \subseteq B \cap K \supsetneq A = B \cap B(\Delta)$. So, B is also standard by maximality of A. So, every Borel subalgebra of L is standard. □